Subdivision schemes with general dilation in the geometric and nonlinear setting

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Abstract

We establish results on convergence and smoothness of subdivision rules operating on manifold-valued data which are based on a general dilation matrix. In particular we cover irregular combinatorics. For the regular grid case results are not restricted to isotropic dilation matrices. The nature of the results is that intrinsic subdivision rules which operate on geometric data inherit smoothness properties of their linear counterparts.

Key words: Nonlinear subdivision, general dilation matrix, irregular vertex

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1. Introduction

The theory of subdivision schemes, with its relations to computer graphics, geometric design, and approximation theory, has grown into a huge body of research. It is mostly concerned with the properties of linear subdivision schemes. There are, however, manifold-valued data which are not directly accessible by linear subdivision rules. In recent years it turned out that *geometric* subdivision, which deals with this kind of geometric data, can be systematically analyzed with regard to convergence, smoothness, and other properties. This line of research was originally proposed by D. Donoho [4, 22]. Wallner and Dyn [24], using the so-called method of proximity, viewed geometric (and necessarily nonlinear) subdivision rules as perturbations of linear ones. They show convergence and C^1 smoothness in the univariate case. Taking this point of view, higher order smoothness results were obtained by subsequent papers such as [23, 26, 6, 27]. Multivariate results, also including irregular combinatorics, are as yet only known for schemes based on dilation matrices which are multiples of the identity [5, 25].

In this paper we are interested in subdivision rules with general dilation operating on geometric data, which includes irregular combinatorics and, in the regular grid case, non-isotropic dilation matrices. Previous work on linear schemes is, for instance, [11, 12, 9] and [16, 15] for regular and irregular combinatorics, respectively. The data we are interested in are points in surfaces and Riemannian manifolds, Lie groups, and symmetric spaces. Examples include positions of a rigid body in space which occur in flight recorder data, or positive definite symmetric matrices which occur in diffusion-tensor imaging. Subdivision rules which in the linear case are mostly defined in terms of averages are modified so as to operate on this kind of data.

This article is organized as follows: Section 2 treats the regular grid case. We establish the setup and formulate our results. In Section 3 we do the same for meshes with irregular combinatorics. All proofs are collected in Section 4.

2. Setup and results for the regular grid case

This section treats subdivision on regular grids based on general dilation matrices. In Section 2.1 we collect the necessary information from linear subdivision, whereas Section 2.2 presents subdivision rules processing geometric data. Section 2.3 contains our results on convergence and smoothness of nonlinear rules for general dilation matrices. Their proofs are given in Section 4.1.

2.1. Linear theory

A linear subdivision scheme is classically given by its mask a, which is a finitely supported sequence $a : \mathbb{Z}^d \to \mathbb{R}$ and an integer dilation matrix $M \in \mathbb{Z}^{d\times d}$ which means that $\lim_{n\to\infty} M^{-n} = 0$. Its action on vector-valued data $p : \mathbb{Z}^d \to V$ is defined by

$$Sp(\alpha) = S_{a,M}p(\alpha) = \sum\nolimits_{\beta\in\mathbb{Z}^d} a(\alpha - M\beta)p(\beta). \tag{2.1}$$

We require that the mask is normalized by $\sum a(\alpha) = |\det M|$. It is well known that uniform convergence for arbitrary bounded data depends only on convergence for input data $\delta_0 : \mathbb{Z}^d \to \mathbb{R}$ which is 1 at 0 and vanishes elsewhere, and that the limit function ϕ associated with this delta sequence is a refinable function which satisfies

$$\phi(x) = \sum\nolimits_{\alpha\in\mathbb{Z}^d} a(\alpha)\phi(Mx - \alpha).$$

All limit functions have the smoothness of the refinable function ϕ, since for input p the corresponding limit function can be written as

$$p * \phi = \sum\nolimits_{\alpha\in\mathbb{Z}^d} p(\alpha)\phi(\cdot - \alpha). \tag{2.2}$$

Smoothness of limits is usually associated with the spectral quantity

$$\rho_k(a, M) = \max_{\mu \in \mathbb{N}_0^d,\ |\mu|=k} \Big(\lim_{n\to\infty} (\|\nabla^\mu S^n \delta_0\|_\infty)^{1/n} \Big). \tag{2.3}$$

This quantity is used to define the smoothness index of a scheme later on. It measures the contraction rate of the k-th order differences of the data produced by the subdivision scheme for input δ_0. The backward difference operator ∇^μ for a multi-index $\mu = (\mu_1, \dots, \mu_d)$ which is used here operates on data p as follows:

$$\nabla^{(\mu_1, \dots \mu_d)} p = (\nabla_{e_1})^{\mu_1} \circ \dots \circ (\nabla_{e_d})^{\mu_d} p \quad \text{with} \quad (\nabla_x p)(y) = p(y) - p(y - x).$$

Here $e_1, \dots, e_d$ are the canonical basis vectors in $\mathbb{R}^d$, and we use the notation $|\mu| = \mu_1 + \dots + \mu_d$.

For the analysis of (2.3) it is required that the scheme satisfies *sum rules*. We say that $S_{a,M}$ satisfies sum rules of order k, if for every polynomial q with $\deg(q) < k$ we have

$$\sum_{\beta\in\mathbb{Z}^d} a(\alpha + M\beta) q(\alpha + M\beta) = \sum_{\beta\in\mathbb{Z}^d} a(M\beta) q(\beta), \qquad \text{for all} \quad \alpha \in \mathbb{Z}^d.$$

There are the following results concerning the smoothness of limits [9].

Theorem 2.1. *Assume the subdivision scheme $S_{a,M}$ has maximal sum rule order k and that the eigenvalues of M are ordered by magnitude of modulus:*

$$\mathrm{spec}(M) = \{\lambda_{\min}, \dots, \lambda_{\max}\}.$$

We define the smoothness index of the scheme by

$$\nu(a, M) = -\frac{\log \rho_k(a, M)}{\log |\lambda_{\max}|}. \tag{2.4}$$

Then the scheme converges if and only if $\nu(a, M) > 0$, and the critical Hölder index of the limit functions is at least $\nu(a, M)$. Further, for $l = 1, \dots, k-1$, we have the equality

$$\rho_l(a, M) = \max(\rho_k(a, M), |\lambda_{\min}|^{-l}). \tag{2.5}$$

The smoothness of the limit functions is measured by their membership in the spaces of the Hölder-Zygmund scale Lip_γ : We define the critical Hölder index of a function f by

$$\nu(f) = \sup\{\gamma : f \in \mathrm{Lip}_\gamma\}.$$

Our terminology regarding Hölder-Zygmund functions is as follows: For $\gamma > 0$ and an integer $k > \gamma$ we consider the seminorm

$$|f|_{\mathrm{Lip}\,\gamma,k} = \inf\{c > 0 \mid \forall\, h \in \mathbb{R}^d \text{ with } \|h\| < h_0 : \|(\nabla_h)^k f\|_\infty < c\|h\|^\gamma\},$$

where h_0 is a positive number. The Hölder-Zygmund space Lip_γ consists of all bounded continuous functions where the norm $\|f\|_{\mathrm{Lip}_\gamma,k} = \|f\|_\infty + |f|_{\mathrm{Lip}_\gamma,k}$ is finite. It is well known that these spaces do not depend on the choice of k and h_0, and that the corresponding norms are equivalent. For more information on Hölder-Zygmund spaces we refer to [21], where they are considered as special instances of Besov spaces and are denoted by $B^\gamma_{\infty,\infty}$.

There is an important special case which implies equality of the smoothness index of the scheme $\nu(a, M)$ and the Hölder index $\nu(\phi)$ of the refinable function; this is when M is *isotropic* and $S_{a,M}$ is *stable*. M is isotropic if M is $\mathbb{C}$-diagonalizable and all eigenvalues are equal in modulus. $S_{a,M}$ is called stable if the mapping $p \to p * \phi$ is lower bounded.

2.2. Geometric subdivision rules

Geometric subdivision acts on data which lie in surfaces, Riemannian manifolds, matrix groups, and the like. They can be defined whenever a substitute for affine averaging which linear rules are based on is available in the respective geometry. This can be either done by employing substitutes for the basic arithmetic operations used in the definition of linear subdivision rules or by using a definition which transfers properties of an affine average to the geometric setting. In order to explain this, we consider the weighted average $x = \sum a_j x_j$ of points x_j with weights a_j summing up to 1. In Euclidean space the following definitions of x are equivalent:

$$x = y \oplus \sum a_j (x_j \ominus y), \qquad \text{for arbitrary } y, \tag{2.6}$$

$$x = \mathrm{argmin}_y \sum a_j \, \mathrm{dist}(x_j, y)^2, \tag{2.7}$$

$$x \text{ solves } \sum a_j (x_j \ominus x) = 0. \tag{2.8}$$

Here the symbols $\oplus$ and $\ominus$ stand for the ordinary $+$ and $-$ operations in a vector space. In any space where modified versions of the $\oplus$ and $\ominus$ operators are available we may use (2.6) or (2.8) to define an analogue of the linear construction. Note, however, that the choice of the 'base point' y in (2.6) in general influences the result.

In a matrix group (or in a general Lie group) we may define

$$p \oplus v = p \exp(v), \quad q \ominus p = \log(p^{-1} q),$$

using the matrix exponential (or Lie group exponential) [1]. It is known that constructions (2.6) and (2.8) are well defined for input data which are close enough. In general, these constructions are not well defined globally since $q \ominus p$ is not well defined. This is because the matrix exponential is one-to-one and onto only for arguments in some neighborhood of zero. The precise conditions depend on the group under consideration.

In a surface or Riemannian manifold we employ the exponential mapping $\exp_p(v)$ which computes the endpoint of a geodesic line emanating from the point p in direction v and whose length equals $\|v\|$ [2]:

$$p \oplus v = \exp_p(v), \quad q \ominus p = \exp_p^{-1}(q).$$

It is known that in this case construction (2.6) is well-defined for close enough input data and base point y nearby. Due to the availability of a metric in a Riemannian manifold additionally (2.7) can be employed. This definition transfers the minimizing property of an affine average to the Riemannian case. In fact, definitions (2.8) and (2.7) (the Riemannian center of mass) are well-defined and do actually coincide for input data which are close enough. The precise meaning of 'close enough' depends on the sectional curvatures of the manifold in question [13]. With these preparations, we define the *log-exp analogue* of a subdivision rule $S_{a,M}$ in a surface or Riemannian manifold by

$$(Tp)(\alpha) = q(\alpha) \oplus \Big(\sum_{\beta \in \mathbb{Z}^d} a(\alpha - M\beta)(p(\beta) \ominus q(\alpha)) \Big), \tag{2.9}$$

where $q(\alpha)$ is a base point which lies close to those data items $p(\beta)$ which contribute to the resulting data item $(Tp)(\alpha)$. Secondly, we define the *intrinsic mean analogue* by

$$(T'p)(\alpha) = \arg\min_q \sum_{\beta \in \mathbb{Z}^d} a(\alpha - M\beta) \operatorname{dist}(p(\beta), q)^2. \tag{2.10}$$

Note that if we choose $q(\alpha) = (T'p)(\alpha)$ as a base point in (2.9), then we get $T = T'$. This can be easily seen from the fact that the constructions (2.7) and (2.8) are equivalent in surfaces and Riemannian manifolds. The fact that $T = T'$ for this special choice of base points is important in the analysis since it allows us to treat the intrinsic mean analogue as a special case of the log-exp analogue.

The log-exp analogue was proposed by D. Donoho et al. [4, 22]. Numerical experiments by G. Xie and T. P.-Y. Yu [26] show that log-exp subdivision rules based on the original choice of base-point enjoy the same smoothness as the linear rule they are derived from only up to C^2. In the same paper they suggest a way of choosing the base point to achieve higher order smoothness:

$$(Tp)(\alpha) = (Qp)(\alpha) \oplus \Big(\sum_{\beta \in \mathbb{Z}^d} a(\alpha - M\beta)(p(\beta) \ominus (Qp)(\alpha)) \Big), \tag{2.11}$$

where Q is an auxiliary interpolatory subdivision scheme analogous to a linear scheme with sufficiently high polynomial reproduction. The choice of base points is also the topic of [7].

Last but not least, there is a somewhat different kind of analogue, the so-called *projection analogue*, which works whenever the manifold N the data live in is embedded into an ambient vector space. Examples are embedded surfaces in Euclidean space, or matrix groups. The output of a linear subdivision scheme S applied to data in the submanifold N is projected back to N. The corresponding geometric scheme has the form

$$T = P \circ S. \tag{2.12}$$

The proper notion of a projection mapping P is the following: If the input data p, with values in N, are dense enough, then Sp does not lie too far from N. So it is sufficient that P is defined in an ϵ-neighborhood U of N. It is required that P is a sufficiently smooth mapping with $P \circ P = P$ and $P(U) \subset N$. Examples are closest point projections or gradient flows [6].

2.3. Results for the regular grid case

In this part we transfer properties of linear schemes to nonlinear schemes, using the method of proximity. The proofs are given in Section 4.1. Results of this type have been obtained by [24, 23, 26] in the univariate case, and by [5] in the multivariate case for standard dilation matrices which are a multiple of the identity matrix. Our method of proof is not via derived schemes as in the above mentioned references. This has to do with the problems derived schemes exhibit in case of general dilation (as already observed by [20]).

The fact that subdivision is well-defined only for dense enough data entails considerable technicalities in the proofs. The exact formulation of the *proximity* between a nonlinear scheme and the linear scheme it is derived from is similarly technical. We introduce the following notions: For a subset N of Euclidean space and a positive real number σ, we consider the class $P_{N,\sigma}$ of σ-dense data which lie in N :

$$P_{N,\sigma} = \Big\{ p \in l^\infty(\mathbb{Z}^d, N) \;\Big|\; \|\nabla_{e_i} p\|_\infty \leq \sigma \text{ for all canonical basis vectors } e_i \Big\}.$$

Typically N is a surface in Euclidean space or some open set in Euclidean space obtained as image of a chart. Further, we consider the quantity

$$\Omega_j(p) = \sum_{\gamma \in \Gamma_j} \prod_{i=1}^{j} \sup_{|\mu|=i} (\|\nabla^\mu p\|_\infty)^{\gamma_i}, \quad \text{where} \quad \Gamma_j = \{\gamma \in \mathbb{N}_0^j \mid \gamma_1 + 2\gamma_2 + \cdots + j\gamma_j = j+1\}. \tag{2.13}$$

For illustration, consider the cases $j = 1$ and $j = 2$:

$$\Omega_1(p) = \sup_{|\mu|=1} \|\nabla^\mu p\|^2, \quad \Omega_2(p) = \sup_{|\mu|=1} \|\nabla^\mu p\|^3 + \sup_{|\mu|=1} \|\nabla^\mu p\| \sup_{|\mu|=2} \|\nabla^\mu p\|.$$

Using this notation, we define proximity between subdivision rules S, T which operate on data living in a Euclidean vector space.

Definition 2.2. *Subdivision rules S and T obey proximity inequalities of order k in the domain $P_{N,\sigma}$ if there is a constant $C > 0$ such that, for all $p \in P_{N,\sigma}$,*

$$\sup_{|\mu|=j-1} \|\nabla^\mu (Sp - Tp)\|_\infty \leq C\Omega_j(p) \quad \text{for } j = 1, \ldots, k. \tag{2.14}$$

This definition has already been successfully employed in [24, 23, 5, 26]. It turns out that also in our setting, allowing dilation matrices to be arbitrary, we can use it to obtain convergence and smoothness of T. The result below concerning convergence of T is rather technical because we must specify the domains our data lie in, and we have to guarantee that T is defined for all intermediate subdivided data.

Definition 2.3. *A subdivision scheme T is called convergent for input data p if $T^n p$ is well-defined for all n, and if there is a uniformly continuous function f_p such that*

$$\|f_p(M^{-i}\cdot) - T^i p\|_{l^\infty(\mathbb{Z}^d)} \to 0 \qquad \text{as } i \to \infty.$$

Here f_p is sampled on $M^{-i}\mathbb{Z}^d$ and a sequence on $\mathbb{Z}^d$ is generated from this sample by the change of coordinates $\alpha \to M^i \alpha$.

Theorem 2.4. *Consider a convergent linear subdivision rule $S_{a,M}$ which is in first order proximity with the subdivision rule T w.r.t. the class of data $P_{N,\sigma}$. We assume that, for all $p \in P_{N,\sigma}$, the subdivided data Tp takes its values in a set $N' \supset N$. Assume further that there is $N'' \subset N$ and $\sigma' > 0$ such that the σ'-neighborhood $U_{\sigma'}(N'')$ obeys $U_{\sigma'}(N'') \cap N' \subset N$. Then the subdivision rule T converges for bounded data p in $P_{N'',\sigma''}$, for some $\sigma'' > 0$. Furthermore, using the notation of Definition 2.3*

$$T^n p * \phi(M^n \cdot) \to f_p \qquad \text{as uniformly continuous functions.}$$

Since we consider quite general sets N in this theorem, we have to assume the existence of the set N'' with the above properties. However, if, for example, N is a ball of radius r, then N'' can be chosen as the ball with the same center and radius $r - \sigma'$. Then, for this particular choice of N, the theorem says that, if S and T fulfill proximity conditions w.r.t. $P_{N,\sigma}$, then T converges for dense enough input in the smaller ball N''.

Theorem 2.5. *Assume that the linear subdivision rule $S_{a,M}$ has maximal sum rule order k and that it is in k-th order proximity with a subdivision rule T w.r.t. to some domain $P_{N,\sigma}$ of σ-dense data. If T converges for input data p, then the limit f_p is a Lip_γ function for all $\gamma < \nu(a, M)$.*

The main reason for deriving these theorems is that they apply to the geometric subdivision rules introduced above: When transferring geometric data into $\mathbb{R}^n$ by means of some coordinate representation, we obtain nonlinear subdivision rules which operate in $\mathbb{R}^n$. Knowing that these resulting rules are in proximity to linear rules, we conclude:

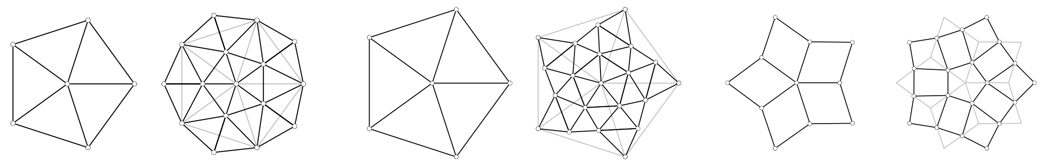
Theorem 2.6. *The previous theorems regarding convergence and smoothness apply to geometric subdivision rules which are the log-exp analogue or the intrinsic mean analogue of the linear rule $S_{a,M}$. In the log-exp case, the choice of base points must follow [7] (of which* (2.11) *is a special case). Furthermore, they apply to the projection analogue, where for the smoothness result it is required that the projection mapping is C^{k+1}.*

Corollary 2.7. *If the linear subdivision scheme $S_{a,M}$ is stable and M is isotropic, the geometric subdivision schemes mentioned in Theorem 2.6 produce limits f_p whose smoothness index $\nu(f_p)$ is at least as high as the smoothness index $\nu(\phi)$ of the refinable function ϕ of $S_{a,M}$. In particular, if $S_{a,M}$ produces C^k limits, then its geometric analogues also produce C^k limits.*

Proofs of the above statements are given in Section 4.1.

3. Setup and results in case of irregular combinatorics.

This section extends the convergence and C^1 smoothness results of the previous section to nonlinear geometric subdivision on polyhedral meshes with possibly irregular combinatorics. In Section 3.1 we briefly recall some basic notions of subdivision on polyhedral meshes. For more information, we refer the reader to [3] and the references therein. Section 3.2 gives a precise definition of parametric convergence near combinatorial singularities for polyhedral meshes, which we need for the formulation of our results. In Section 3.3 we formulate assumptions on linear schemes which allow us to deduce convergence and C^1 smoothness for nonlinear schemes in proximity to them. Section 3.4 explains how the geometric schemes of Section 2.2 are defined in the setting of polyhedral meshes. Finally, in Section 3.5 we formulate our results. The respective proofs are given in Section 4.2.



$\sqrt{3}$ refinement. $\sqrt{7}$ refinement. Dual $\sqrt{2}$ refinement.

Figure 1: Different ways of refining the combinatorics near an extraordinary vertex/face.

3.1. Subdivision schemes on irregular combinatorics

Here we set up subdivision on two-dimensional polyhedral meshes with possibly irregular combinatorics. A mesh is defined by its combinatorics consisting of sets of vertices, edges and faces and a positioning function h assigning each vertex its geometric position. A subdivision scheme refines both the combinatorics and the geometric positions of the vertices. For linear rules the refinement of the geometric positions is usually described by so-called *stencils* $\alpha_{v,w}$: The position $h_1(w) = (Sh_0)(w)$ of a vertex w of the refined combinatorics is given by

$$h_1(w) = \sum_v \alpha_{v,w} h_0(v), \quad \text{where} \quad \sum_v \alpha_{v,w} = 1, \tag{3.15}$$

and $h_0(v)$ are positions of vertices in the initial combinatorics. It is assumed that $\alpha_{v,w} \neq 0$ for only finitely many v and that the corresponding v are combinatorially near to the vertex w. Different types of combinatorial refinement are depicted in Figure 1.

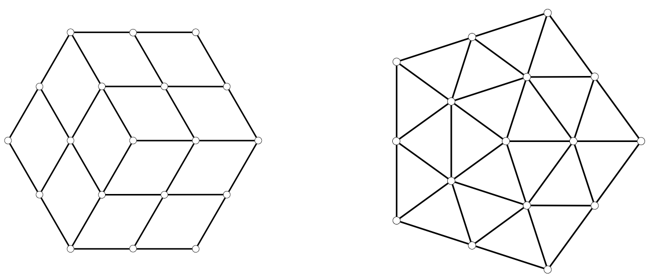


Figure 2: k-regular meshes for the quad case (*left*) and the triangular case (*right*).

It is a typical feature of subdivision with local rules that after a few rounds of subdivision, the combinatorial singularities (*extraordinary* faces or vertices) become well isolated. It is therefore no loss of generality to restrict analysis to the case of so-called k-regular meshes (Figure 2) whose combinatorics posses a single face or vertex of valence k in the center, surrounded by a regular mesh. This can be a quad mesh (or a triangular mesh) where regularity means faces and vertices of valences 4 and 4, respectively (or 3 and 6, respectively).

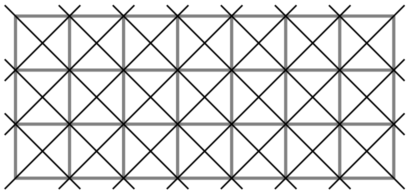


Figure 3: Rotation of the regular lattice about $\pm 45°$ by $\sqrt{2}$-refinement.

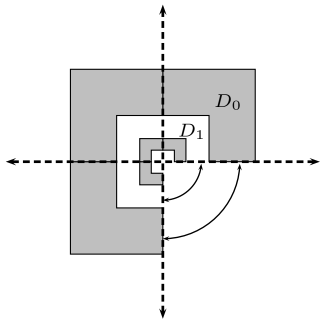


Figure 4: Splitting of the domain D into rings for refinement based on the dilation matrix $2I$.

The linear subdivision rules considered in this section are, in the regular part of a mesh, given by a subdivision operator of the form $S_{a,M}$ with a finitely supported mask a and a dilation matrix M. The information encoded in the mask yields the stencils for the regular parts of the mesh, whereas near singularities modified averaging rules are employed. In this section on irregular combinatorics we consider only classes of schemes based on *isotropic* dilation matrices which are associated with a rotation of the regular quadrilateral lattice in the plane or the regular triangular lattice in the plane, respectively.

For $\sqrt{2}$-schemes and $\sqrt{3}$-schemes the corresponding angle is $\pm 45°$ (see Figure 3), in case of $\sqrt{7}$-schemes it is $\pm \arctan(\sqrt{3}/5)$ which is not a rational multiple of π [15]. For a discussion of different ways of refining the combinatorics, desired properties in application in geometric modeling, and attempts toward a classification of subdivision schemes we refer to [3] and the references therein. Another reference is [15].

3.2. Definition of convergence

Our objective is to derive convergence and smoothness results for nonlinear schemes acting on meshes with irregular combinatorics. To that end we first define a parametric notion of convergence near the singularity in a k-regular mesh. Consider Definition 2.3. There are two notions in this parametric definition of convergence in the regular mesh case which are not a priori determined near the singularity: The grid $\mathbb{Z}^2$ with its refinements $M^{-n}\mathbb{Z}^2$, as well as the domain of the limit function $\mathbb{R}^2$. We define substitutes for these two objects for k-regular meshes.

We start with the domain D where the limit function is defined in: We obtain D by cyclically gluing k copies of a sector Ω in the plane with opening angle $90°$ in the quad case (or $60°$ in triangular case), i.e.,

$$D = \Omega \times \mathbb{Z}_k,$$

where $\mathbb{Z}_k$ are the integers modulo k. We refer to Figure 5 for a visualization. The gluing is done as follows: In each sector we have polar coordinates (x, ϕ) where $0 \leq \phi \leq 90°$ $(60°)$. The points $(x, 90°)$ of the first sector and the points $(x, 0°)$ of the second sector are identified, and so on, where the points $(x, 90°)$ in the k-th sector and $(x, 0°)$ in the first sector are also identified. In the triangular case, $(x, 90°)$ is replaced by $(x, 60°)$. In this way we obtain polar coordinates on D where angles vary between $0°$ and $k\ 90°$ ($k\ 60°$ in the triangular case). For example, a point in D with polar coordinates $(x, 110°)$ comes from the second sector and has angle $20°$ in that sector. The domain D is an abstract space which turns into a metric space by defining the distance of points by the length of the shortest path which connects them, with the metric in the single sectors being that of $\mathbb{R}^2$.

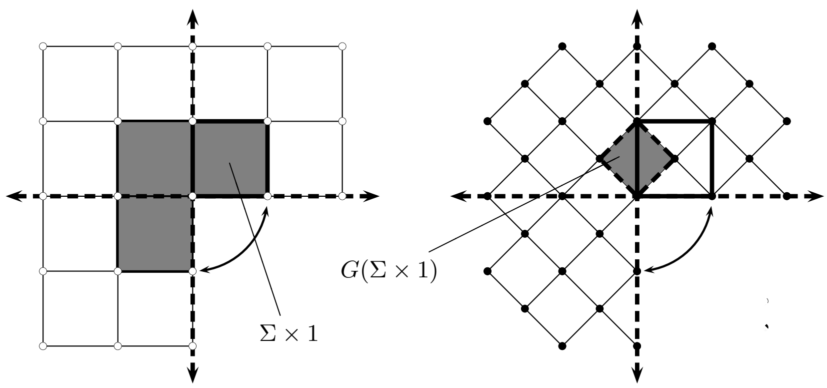


Figure 5: Primal $\sqrt{2}$-subdivision for valence $k = 3$ near a central irregular vertex. Vertex sets V_0 (left) and $V_1 = GV_0$ (right) act as substitutes for $\mathbb{Z}^2$ and $M^{-1}\mathbb{Z}^2$.

Next, we define the substitute for the grid $\mathbb{Z}^2$ and its refinements $M^{-n}\mathbb{Z}^2$, where we consider the primal case in detail. We start with the domain for the initial k-regular mesh. Let Σ be the unit square $[0, 1] \times [0, 1] \subset \Omega$ in the quadrilateral case (the equilateral triangle of length 1 in the triangular case). In the quadrilateral case, there is a quadrangulation of the sector Ω such that each quadrilateral is congruent to Σ. Gluing these sector-wise quadrangulations together we obtain a quadrangulation of D whose vertices define the set V_0 which serves as domain for the initial k-regular mesh; see Figure 5 for a visualization. For the triangular case we proceed analogously by starting with a triangulation of Ω where each triangle is congruent to Σ.

Next we define the domains $V_1, V_2, \ldots$ for the subdivided k-regular meshes which serve as a substitute for the refined grids $M^{-n}\mathbb{Z}^2$. To that end we introduce notions of *dilation* and *rotation* on D : In polar coordinates, dilation by a factor $\lambda > 0$ is given by $(x, \phi) \to (\lambda x, \phi)$; rotation about an angle ψ is given by $(x, \phi) \to (x, \phi + \psi)$. The dilation matrix M now induces a 'similarity transform' $G = G_{m^{-1},\psi}$ with dilation $m^{-1} = |\det M|^{-1}$ and rotation angle ψ which is the same as the rotation angle in the regular case. We define

$$V_i = G^i V_0.$$

The action of a subdivision scheme T on such k-regular input meshes is interpreted in the following way: It transforms data $h : V_n \to \mathbb{R}^d$ at level n to new data $T_n h : V_{n+1} \to \mathbb{R}^d$. We explicitly distinguish the operations on different levels since we find it more convenient for the analysis of nonlinear schemes. We now can define convergence near a singularity:

Definition 3.1. *A subdivision rule T converges on the bounded k-regular mesh $p : V_0 \to \mathbb{R}^d$, if iterated subdivision for input p is well-defined and if there is a uniformly continuous function $f_p : D \to \mathbb{R}^d$ such that*

$$\|f|_{V_i} - T_{i-1,0}p\|_\infty \text{ converges to } 0, \text{ as } i \to \infty.$$

Here $T_{i,l}$ is short for

$$T_{i,l} = T_i \circ \ldots \circ T_l \quad \text{for} \quad i \geq l,$$

and $T_{i,l}$ is the identity if $i < l$. $T_{i-1,0}$ maps data on subdivision level 0 to data on level i performing i steps of subdivision. For the limit we use the notation $T_{\infty,0}p := f_p$.

There is another interesting scheme we would like to incorporate into our framework, namely J. Peters' and U. Reif's simplest subdivision scheme [17] (the mid-edge subdivision scheme of [8]). This scheme is a dual $\sqrt{2}$-scheme, whose dilation matrices correspond to the similarity transforms $G = G_{1/\sqrt{2},\pm\pi/4}$. Except for the choice of the discrete domain V_0, the framework we presented

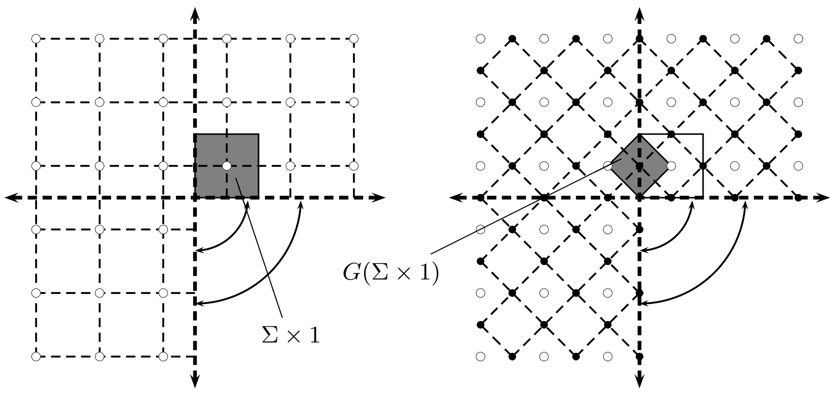


Figure 6: Choice of vertex sets V_0 (left) and $V_1 = GV_0$ (right) for dual $\sqrt{2}$-subdivision near an irregular vertex of valence $k = 3$.

for primal schemes can remain unchanged. Here is how to choose V_0 and the refinements V_i such that the class of dual $\sqrt{2}$-schemes also fits into our framework (see Figure 6):

$$V_0 = ((\tfrac{1}{2}, \tfrac{1}{2}) + \mathbb{N}_0 \times \mathbb{N}_0) \times \mathbb{Z}_k, \quad \text{and} \quad V_i = G^i V_0.$$

The above setup is in the spirit of the one introduced by Reif [19]. However, our setup incorporates more general dilation matrices. Furthermore we have to include some discrete components, namely the sets V_i, which cannot be found in [19] and which we need for the analysis of nonlinear schemes.

3.3. Linear subdivision rules

The first step in the convergence and smoothness analysis is to split the neighborhood of the singularity into so-called rings D_i. The familiar splitting in the case of schemes based on dilation matrix $2I$ is shown in Figure 4. In general this is done as follows (see Figure 7): We start with

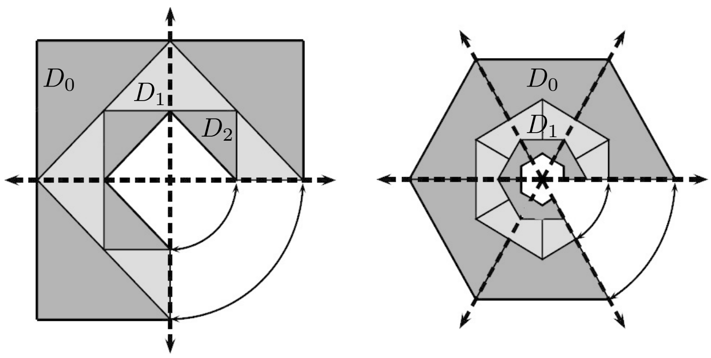


Figure 7: Domains of the limit functions and auxiliary rings D_0, D_1, $D_2, \ldots$ *Left:* Quad-based $\sqrt{2}$ scheme. *Right:* Triangle-based $\sqrt{3}$-scheme.

a certain neighborhood $D' = D'(r)$ of the singular point 0 of the domain D given by

$$D' := r\Sigma \times \mathbb{Z}_k,$$

where r denotes some scaling factor which is explained later on and which should not be confused with the radial component of some polar coordinate. D' is obtained as the union of all copies of $r\Sigma$ in all sectors. Using the similarity transform G of Section 3.2 we obtain rings $D_i = D_i(r)$ for $i = 0, 1, \ldots$ as follows:

$$D_i = G^i D' \setminus G^{i+1} D'.$$

The segments $D_i^j = D_i^j(r)$ and the i-th inner area $D_i' = D_i'(r)$ are defined by

$$D_i^j = G^i(\Omega \times j) \cap D_i, \qquad D_i' = G^i D'. \tag{3.16}$$

Before formulating the assumptions on the schemes we consider, we have to introduce the notion of *control sets*. It is well known that due to the locality of the subdivision rule, the limit function on a bounded set $U \subset D$ only depends on data on some finite subset of V_i on each level i. The smallest such subset is called the control set of U on level i and is denoted by $\mathrm{ctrl}^i(U)$.

We consider linear subdivision rules with smoothness index $\nu_{a,M} > 1$ on regular meshes. By choosing the factor $r > 0$ big enough we can achieve that the control sets $\mathrm{ctrl}^i(D_i^j(r))$ are vertices of a regular connectivity. Then we find, by perhaps enlarging r, a linear mapping (represented by a *square* matrix A) which maps data on the control set of the 0-th inner area $D_0'(r)$, to data on the control set of the first inner area $D_1'(r)$. This square matrix A is called the *subdivision matrix*; it is the basis of convergence and smoothness analysis. This notion is not as general as the corresponding one in the book [18] which is due to our discrete approach.

We impose the following conditions on the subdivision matrix A:

(i) The largest eigenvalue of A equals 1.

(ii) There is a unique pair of complex conjugate subdominant Jordan blocks or a unique pair of real subdominant Jordan blocks with same multiplicity and same eigenvalues.

(iii) We choose one Jordan vector with the highest multiplicity and consider its real part v_1 and imaginary part v_2 (For real subdominant Jordan blocks, we let v_1 and v_2 be two real Jordan vectors with the highest multiplicity.) With these real-valued input data we construct the limit functions $\chi_1 = S_{\infty,0}v_1$, $\chi_2 = S_{\infty,0}v_2$. We assume that the mapping $\chi = (\chi_1, \chi_2) : D \to \mathbb{R}^2$ (the *characteristic map*) is regular and injective on the punctured set $U \setminus \{0\}$, where U is a neighborhood of 0.

In the monograph [18] the quite natural situation (ii) occurs in case of so-called shift invariant algorithms. Schemes which fulfill all these requirements are the $\sqrt{3}$ and $\sqrt{7}$ schemes of Oswald et al. [16, 15], which include the $\sqrt{3}$ scheme of [14], and the mid-edge subdivision scheme [17]. The following theorem of U. Reif is also valid in the case of our more general dilation matrices as, for example, observed in the papers [16, 15].

Theorem 3.2. *Let S be a linear subdivision scheme fulfilling the assumptions above. For input data p on input level V_0 consider the limit function f_p. Then the map $f_p \circ \chi^{-1}$ is well-defined and C^1 in a neighborhood of 0. For almost all input data the image of f_p is a two-dimensional submanifold of $\mathbb{R}^d$ locally around the limit point $f_p(0)$.*

3.4. Geometric subdivision rules

A geometric rule T analogous to a linear rule S can be obtained by firstly using the same refinement procedure of the combinatorics for T as for S. For S, the calculation of new vertex positions was based on (3.15) which is an affine averaging rule. Secondly, this averaging rule can be modified as explained in Section 2.2 to obtain geometric rules which are intrinsically defined in surfaces, Riemannian manifolds, Lie groups, etc.

3.5. Results for meshes with irregular combinatorics

This section of results deals only with the case of k-regular meshes as described in Section 3.1. This is in fact equivalent to the case of general combinatorics since combinatorial singularities become well isolated after a few rounds of subdivision.

By suitable coordinate representations of the (dense enough) geometric data, each geometric subdivision scheme can be viewed as a (nonlinear) scheme T acting on data with values in $\mathbb{R}^n$. To compare this scheme T with a linear scheme S we need the following local proximity inequality.

Definition 3.3. *Combinatorially equivalent subdivision rules S and T fulfill a local (first order) proximity inequality w.r.t. a set $P_{N,\sigma}$ of σ-dense data if there is a constant $C > 0$ such that for all data $p \in P_{N,\sigma}$*

$$\|Sp(w) - Tp(w)\| \leq C \sup\{\|p(x) - p(y)\| : x, y \in \operatorname{supp}(\alpha_{\cdot,w})\}^2. \tag{3.17}$$

Here $\operatorname{supp}(\alpha_{\cdot,w}) = \{v : \alpha_{v,w} \neq 0\}$ is the support of the stencil $\alpha_{v,w}$ which are those vertices v which contribute to the calculation of $Sp(w)$. It is actually not difficult to generalize the result of [25] concerning convergence to the case of more general dilation matrices:

Theorem 3.4. *If S is a linear convergent scheme according to Section 3.3 which is in proximity with the (nonlinear) scheme T, then T converges for dense enough input data.*

The precise statement is analogous to the regular grid case. It is given in Section 4.2 together with its proof. As to smoothness, we have the following result:

Theorem 3.5. *Assume that the linear subdivision scheme S and and the scheme T fulfill the local proximity inequality* (3.17) *w.r.t. some class $P_{N,\sigma}$ of σ-dense input. Then the limit of subdivision using T is continuously differentiable w.r.t. the characteristic parametrization. More precisely, the function $T_{\infty,0}p_0 \circ \chi^{-1}$ is well-defined and C^1 in a neighborhood of the (extraordinary) point $\chi(0)$.*

Knowing that the geometric analogues considered in Section 2.2 fulfill local first order proximity inequalities with the linear scheme they are derived from, we conclude:

Corollary 3.6. *Theorems 3.4 and 3.5 concerning convergence and smoothness apply to the following kinds of geometric subdivision rules: the log-exp analogue, the intrinsic mean analogue, and the projection analogue.*

The proofs of these statements can be found in Section 4.2.

4. Proofs

Concerning the constants in the proofs of this section we employ the following conventions: Whenever it is possible, we use a generic constant C which can change from line to line. However, there are some proofs where it is necessary to distinguish constants. For such constants we do not use the symbol C.

4.1. Proofs for the regular grid case

In this section we prove Theorem 2.4 and Theorem 2.5. The first two statements below are auxiliary statements which concern linear subdivision. Lemma 4.1 starts from (2.3) and establishes inequality (4.18). The main point here is that differences are incorporated in the right-hand side of (4.18) and that general input data are considered, which will be important for the analysis of nonlinear schemes. We did not find this statement in the literature, even it is possibly already known.

Lemma 4.1. *Assume that $S = S_{a,M}$ is a linear convergent subdivision rule which satisfies sum rules of order k. Then for every $s > 1$ there is $C \geq 1$ such that, for all $p \in l^\infty(\mathbb{Z}^d)$ and all $n \in \mathbb{N}_0$,*

$$\sup_{|\mu|=k} \|\nabla^\mu S^n p\|_\infty \leq C(\rho_k s)^n \sup_{|\mu|=k} \|\nabla^\mu p\|_\infty. \tag{4.18}$$

Proof. By definition of ρ_k (2.3) there is a constant $C > 0$ such that, for $s > 1$,

$$\|\nabla^\mu S^n \delta_0\|_\infty \leq C(\rho_k s)^n \quad \text{for all multiindices } \mu \text{ with } |\mu| = k. \tag{4.19}$$

The constant C depends on the choice of s but not on the exponent n. We use the notation $l(\mathbb{Z}^d)$ for the space of sequences on $\mathbb{Z}^d$. We consider the mapping

$$p \mapsto \{\nabla^\mu S^n p\}_{|\mu|=k} \tag{4.20}$$

from $l(\mathbb{Z}^d)$ to $l(\mathbb{Z}^d)^r$, where $r = \binom{k+d-1}{k}$ is the number of different multiindices with $|\mu| = k$. This mapping is linear. We show that this mapping only depends on the k-th order differences of the input p, i.e., it only depends on $\{\nabla^\mu p\}_{|\mu|=k}$: Since S satisfies sum rules of order k, S leaves the set of samples of polynomials of degree lower than k invariant (see [12], Theorem 5.2). A sample of a polynomial p with $\deg(p) < k$ is characterized by the vanishing of all differences of order k, i.e., $\nabla^\mu p = 0$ for all μ with $|\mu| = k$. These two observations guarantee that the property $\nabla^\mu p = 0$ for all multiindices μ with order k implies $\nabla^\mu S p = 0$ whenever $|\mu| = k$. This implies that the mapping (4.20) only depends on the k-th order differences of p.

With these observations at hand we use the locality of the subdivision scheme S and construct a scenario which allows us to apply the uniform bounded principle which then yields (4.18). To that end, we consider the 'discrete simplex' $T = \{\alpha \in \mathbb{N}_0^d : |\alpha| < k\}$, and choose $N > 2k$ so large that the limit function of subdivision on $[-1,1]^d$ for input p only depends on the values of p on $Q = \{-N, \dots, N\}^d$ (It is well known that for finitely supported masks such an N exists.) We start with (possibly unbounded) data $p \in l(\mathbb{Z}^d)$ and find $p' \in l(\mathbb{Z}^d)$ with

$$\nabla^\mu p = \nabla^\mu p' \ (\mu \text{ with } |\mu| = k) \quad \text{and} \quad p'|_T = 0. \tag{4.21}$$

This is done by finding a polynomial with degree lower than k which agrees with p on T and subtracting it from p. We use the notation $l(A)$ for the space of sequences on $\mathbb{Z}^d$ vanishing outside $A \subset \mathbb{Z}^d$. We consider the projection operator $P : l(\mathbb{Z}^d \setminus T) \to l(Q \setminus T)$, which sets values outside Q to 0. We get a constant C which is independent of p such that

$$\sup_{|\mu|=k} \|\nabla^\mu P p\|_\infty \leq C \sup_{|\mu|=k} \|\nabla^\mu p\|_\infty.$$

We consider the family of operators

$$(\rho_k s)^{-n} \nabla^\mu S^n : l(Q \setminus T) \to l^\infty(\mathbb{Z}^d),$$

indexed by the mutiindex μ and the exponent n. This family is bounded on any sequence q. The principle of uniform boundedness yields a constant C, independent of q, n, and μ, such that

$$\sup_{|\mu|=k} \|\nabla^\mu S^n q\|_\infty \leq C(\rho_k s)^n \sup_{|\mu|=k} \|\nabla^\mu q\|_\infty$$

for $q \in l(Q \setminus T)$.

We consider general $p \in l^\infty(\mathbb{Z}^d)$ and choose a sequence p' according to (4.21) and define $q \in l(Q \setminus T)$ by $q = Pp'$. Then we use the above estimates to get

$$\sup_{|\mu|=k} \|\nabla^\mu S^n q\|_\infty \leq C(\rho_k s)^n \sup_{|\mu|=k} \|\nabla^\mu p\|_\infty.$$

Furthermore, for any multiindex μ of order k, we have that $\nabla^\mu S^n q = \nabla^\mu S^n p$ on $\{-k, \dots, k\}^d$. In view of the translation invariance of S, this implies (4.18). $\square$

The next statement also concerns linear subdivision. Its purpose is to estimate Lip-seminorms of the limit functions by differences of the data.

Proposition 4.2. *Assume that $S_{a,M}$ is a linear convergent subdivision operator which has maximal sum rule order k. Then for every γ which is smaller that the smoothness index $\nu(a, M)$, the* *mapping $p \mapsto p * \phi(M^m \cdot)$ of data on level m to limit functions is a bounded linear operator from $l^\infty(\mathbb{Z}^d)$ to Lip_γ for every input level m. The growth of the bounds of the Lip_γ-seminorms in m can be estimated by differences of input data as follows: For all $s > 1$ there is $C \geq 1$ such that*

$$|p * \phi(M^m \cdot)|_{\mathrm{Lip}_\gamma, k} \leq C(|\lambda_{\max}| s)^{m\gamma} \sup_{|\mu|=k} \|\nabla^\mu p\|_\infty, \tag{4.22}$$

where C is independent of m, and $\lambda_{\max}$ is an eigenvalue of M of greatest modulus.

Proof. Since the refinable function ϕ is a Lip_γ function, we have, for every $s > 1$, a constant $C > 0$ such that, for every nonnegative integer m,

$$|\phi(M^m \cdot)|_{\mathrm{Lip}_\gamma, k} \leq C |\lambda_{\max}|^{m\gamma} s^m.$$

As a consequence, the Lip_γ-seminorm for arbitrary bounded input data p can be estimated by

$$|p * \phi(M^m \cdot)|_{\mathrm{Lip}_\gamma, k} \leq C |\lambda_{\max}|^{m\gamma} s^m \|p\|_\infty.$$

This is due to the compact support of ϕ. Since S satisfies k-th order sum rules, $q * \phi$ is a polynomial with $\deg(q * \phi) < k$ for any sample q of a polynomial of degree lower than k (see e.g. the discussion around Theorem 2.1 in [12]). Therefore, the directional difference $\nabla_y^k p * \phi$ of the limit function for input p only depends on the k-th order differences $\{\nabla^\mu p\}_{|\mu|=k}$.

We use the notation of the proof of Lemma 4.1, and define, for $p \in l^\infty(\mathbb{Z}^d)$, the sequence $q \in l(Q \setminus T)$ by $q = Pp'$, where p' is chosen according to (4.21). Then in the cube $[-1, 1]^d$, the limit functions $p * \phi$ and $q * \phi$ are equal. If we consider the smaller cube $[-1/2, 1/2]^d$, we find a step size $h > 0$, such that the difference $\nabla_y^k p * \phi$ and $\nabla_y^k q * \phi$ agree for all vectors $y \in \mathbb{R}^d$ with $\|y\| \leq h$.

We consider the family of operators $l(Q \setminus T) \to \mathrm{Lip}_\gamma$,

$$q \mapsto |\lambda_{\max}|^{-m\gamma} s^{-m} q * \phi(M^m \cdot),$$

which is indexed by the exponent m. This family is bounded on every sequence $q \in l(Q \setminus T)$. Therefore, the principle of uniform boundedness yields a constant $C > 0$, which is independent of q and m such that

$$|q * \phi(M^m \cdot)|_{\mathrm{Lip}_\gamma, k} \leq C |\lambda_{\max}|^{m\gamma} s^m \sup_{|\mu|=k} \|\nabla^\mu p\|_\infty.$$

This yields (4.22). ∎

The next lemma consists of Equations (4.23) and (4.24). The estimate (4.23) establishes a certain contractivity of the nonlinear scheme T. A similar estimate is also an important intermediate step in all previous smoothness proofs. In addition, we obtain the important estimate (4.24) which is central in the proof of Theorem 2.5.

Lemma 4.3. *Assume that $S_{a,M}$ is a linear convergent subdivision scheme with maximal sum rule order k. Assume furthermore that $S_{a,M}$ and the (nonlinear) scheme T fulfill k-order proximity conditions w.r.t. some class $P_{N,\sigma}$ of σ-dense input.*

Then for any $s > 1$, we can find $C > 0$ and $\sigma'' > 0$ such that the following is true: For input $p \in P_{N,\sigma''}$, for which we assume that $T^n p$ is defined for all n and that $T^n p \in P_{N,\sigma}$ for all n, for any $j \in \{1, \dots, k\}$ we have the inequality

$$\sup_{|\mu|=j} \|\nabla^\mu T^n p\|_\infty \leq C \max(\rho_k, |\lambda_{\min}|^{-j})^n s^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty, \tag{4.23}$$

where C is independent of p. In particular there is $L > 0$ with

$$\Omega_j(T^n p) \leq L(\rho_j \rho_1 s)^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2. \tag{4.24}$$

Proof. If the statement holds for some $s > 1$, it obviously holds for any $s' > s$. So we can fix $s > 1$ such that $\rho_j s < 1$ for all $j = 1, \dots, k$. For every $j \in \{1, \dots, k\}$ there is, by Lemma 4.1, a constant C'_j (dependent on s) such that

$$\sup_{|\mu|=j} \|\nabla^\mu S^n p\|_\infty \leq C'_j (\rho_j s)^n \sup_{|\mu|=j} \|\nabla^\mu p\|_\infty.$$

We let $C' = \max_{1 \leq j \leq k} C'_j$. Furthermore, we denote the proximity constants from (2.14) by C_P.

For the next estimate, we consider $j \in \{1, \dots, k\}$ and a multiindex μ of order j. We apply Lemma 4.1 and (2.14) in order to obtain , for every $n \in \mathbb{N}$, the estimate

$$\begin{aligned}
\|\nabla^\mu T^n p\|_\infty &\leq \sum_{l=0}^{n-1} \|\nabla^\mu S^l (T - S) T^{n-l-1} p\|_\infty + \|\nabla^\mu S^n p\|_\infty \\
&\leq 2C' \sum_{l=0}^{n-1} \rho_j^l s^l \sup_{|\eta|=j-1} \|\nabla^\eta (T - S) T^{n-l-1} p\|_\infty + C' \rho_j^n s^n \sup_{|\mu|=j} \|\nabla^\mu p\|_\infty \\
&\leq 2C' C_P \sum_{l=0}^{n-1} \rho_j^l s^l \Omega_j (T^{n-l-1} p) + C' \rho_j^n s^n \sup_{|\mu|=j} \|\nabla^\mu p\|_\infty.
\end{aligned} \tag{4.25}$$

Recall that by Theorem 2.1, $\rho_m = \max(\rho_k, |\lambda_{\min}|^{-m})$ for $m < k$. We use induction on 'the order of differences' j to show (4.23) and start with the case $j = 1$. We show (4.23) for the case $j = 1$ for the constants

$$C = C_1 := 2C' \quad \text{and} \quad \sigma'' = \sigma''_1 := \min(\sigma, \frac{\rho_1 s(1 - \rho_1 s)}{8C'^2 C_P}, 1). \tag{4.26}$$

To that end, we perform induction on the subdivision level n. The case $n = 0$ is clear, since $C' \geq 1$. As to general n assume that (4.23) holds for all smaller values than n (still, $j = 1$). Observing that we set $C = 2C'$ in (4.26), we have

$$\Omega_1(T^{n-l-1}p) = \sup_{|\mu|=1} \|\nabla^\mu T^{n-l-1}p\|_\infty^2 \leq 4C'^2(\rho_1 s)^{2n-2l-2} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 \tag{4.27}$$

by the induction hypothesis. This implies

$$\sup_{|\mu|=1} \|\nabla^\mu T^n p\|_\infty \leq C' \rho_j^n s^n \Big(8C'^2 C_P \big(\sum_{l=0}^{n-1} (\rho_1 s)^{n-l-2}\big) \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty + 1\Big) \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty. \tag{4.28}$$

Applying the geometric series, we get

$$\sum\nolimits_{l=0}^{n-1} (\rho_1 s)^{n-l-2} \leq (\rho_1 s)^{-1}(1-\rho_1 s)^{-1}. \tag{4.29}$$

Our choice of σ_1'' in (4.26) implies that

$$\sup\nolimits_{|\mu|=1} \|\nabla^\mu p\|_\infty \leq \sigma_1'' \leq 1/8\ C'^{-2} C_P^{-1} \rho_1 s (1-\rho_1 s). \tag{4.30}$$

Plugging (4.29) and (4.30) into (4.28), we obtain (4.23) for the case $j = 1$.

We assume now that (4.23) is valid for $j-1$ instead of j and perform the induction step. As in the case $j = 1$ we may assume that s is chosen in a way such that $\rho_1 s < 1$, as well as $\rho_j s < 1$. We let

$$s' = s^{1/(j+1)}. \tag{4.31}$$

By the induction hypothesis there is a constant $C_{j-1} > 0$ and a 'denseness constant' σ_{j-1}'' such that, for $m = 1, \ldots, j-1$,

$$\sup_{|\mu|=m} \|\nabla^\mu T^n p\|_\infty \leq C_{j-1}(\rho_m s')^n \sup_{|\mu|=m} \|\nabla^\mu p\|_\infty$$

for all input data $p \in P_{N,\sigma_{j-1}''}$ for which iterated subdivision using T is defined and for which $T^r p \in P_{N,\sigma}$ for all $r \in \mathbb{N}$. We perform induction on n to show (4.23) for the constants

$$C = 2C' \quad \text{ and } \quad \sigma'' = \sigma_j'' = \min(\sigma_{j-1}'', \frac{\rho_1 s((\rho_1 s)^{-1} - 1)\rho_j s}{2DC_P}, 1),$$

where we define the constant D by

$$D = 2C_1 C' + |\Gamma_j| C_{j-1}^{j+1} 2^{j+1}.$$

The choice of D will become clear from the following. The case $n = 0$ is obvious. For the induction step we assume that (4.23) is valid for smaller values than n. There is only one $\gamma \in \Gamma_j$

with $\gamma_j \neq 0$, namely $\gamma = (1, 0, \ldots, 0, 1)$. Using the induction hypothesis its contribution to (2.13) can be estimated as follows:

$$\sup_{|\mu|=j} \|\nabla^\mu T^{n-l-1} p\|_\infty \sup_{|\mu|=1} \|\nabla^\mu T^{n-l-1} p\|_\infty \leq 2C_1 C' (\rho_j \rho_1 s^2)^{n-l-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2.$$

For the other summands $\gamma \in \Gamma_j$ (with $\gamma_j = 0$) we obtain

$$\begin{aligned}
\prod_{i=1}^{j} \sup_{|\mu|=i} \|\nabla^\mu T^{n-l-1} p\|_\infty^{\gamma_i} &\leq \prod_{i=1}^{j} C_{j-1}^{\gamma_i} (\rho_i s')^{\gamma_i (n-l-1)} \sup_{|\mu|=i} \|\nabla^\mu p\|_\infty^{\gamma_i} \\
&\leq C_{j-1}^{j+1} s'^{(j+1)(n-l-1)} \prod_{i=1}^{j} \rho_i^{\gamma_i(n-l-1)} 2^{i\gamma_i} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^{\gamma_i} \\
&\leq C_{j-1}^{j+1} s^{(n-l-1)} 2^{j+1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 \prod_{i=1}^{j} \rho_i^{\gamma_i (n-l-1)}. \qquad (4.32)
\end{aligned}$$

Next, we show the estimate

$$\prod\nolimits_{i=1}^{j} \rho_i^{\gamma_i} \leq \rho_j \rho_1. \qquad (4.33)$$

We distinguish different cases: If $j \leq -\log_{|\lambda_{\min}|} \rho_k$, which means that $\rho_k \leq |\lambda_{\min}|^{-j}$, we apply Theorem 2.1 and obtain that $\rho_i = |\lambda_{\min}|^{-i}$ for all $1 \leq i \leq j$. As a consequence,

$$\prod\nolimits_{i=1}^{j} \rho_i^{\gamma_i} = |\lambda_{\min}|^{-j-1} = \rho_j |\lambda_{\min}|^{-1} \leq \rho_j \rho_1,$$

where the last inequality is also a consequence of Theorem 2.1. This shows (4.33) in case that $j \leq -\log_{|\lambda_{\min}|} \rho_k$. So we can assume that $j > -\log_{|\lambda_{\min}|} \rho_k$, i.e., $\rho_k > |\lambda_{\min}|^{-j}$. If there is some non-zero factor γ_{i_0} such that $i_0 \geq -\log_{|\lambda_{\min}|} \rho_k$, then

$$\prod\nolimits_{i=1}^{j} \rho_i^{\gamma_i} \leq \rho_k \rho_1 = \rho_j \rho_1.$$

This is true since $\rho_{i_0} = \rho_k = \rho_j$ and $\rho_i \leq \rho_1$. If $\gamma_i \neq 0$ only for i smaller than $-\log_{|\lambda_{\min}|} \rho_k$, then

$$\prod\nolimits_{i=1}^{j} \rho_i^{\gamma_i} = |\lambda_{\min}|^{-j-1} \leq \rho_j \rho_1.$$

This shows (4.33). Using the estimate (4.33) in (4.32), we obtain

$$\begin{aligned}
\Omega_j(T^{n-l-1} p) &\leq (2C_1 C' + (|\Gamma_j| - 1) C_{j-1}^{j+1} 2^{j+1}) (\rho_j \rho_1 s^2)^{n-l-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 \\
&\leq D(\rho_j \rho_1 s^2)^{n-l-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2. \qquad (4.34)
\end{aligned}$$

We use (4.25) and (4.34) to obtain

$$\begin{aligned}
\sup_{|\mu|=j} \|\nabla^\mu T^n p\|_\infty &\le 2C' C_P \sum_{l=0}^{n-1} \rho_j^l s^l \Omega_j(T^{n-l-1}p) + C' \rho_j^n s^n \sup_{|\mu|=j} \|\nabla^\mu p\|_\infty \\
&\le 2C' C_P D \sum_{l=0}^{n-1} (\rho_j s)^{n-1} (\rho_1 s)^{n-l-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 + C' \rho_j^n s^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty \\
&\le C' \rho_j^n s^n \Big(2C_P D (\rho_1 s)^{-1} ((\rho_1 s)^{-1} - 1)^{-1} (\rho_j s)^{-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty + 1 \Big) \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty \\
&\le 2C' \rho_j^n s^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty .
\end{aligned}$$

The last inequality is valid since, by the choice of σ_j'', the term in brackets is smaller than 2. So the induction w.r.t. both n and j is complete. Finally, the statement (4.24) is shown by (4.27) and (4.34). □

With these preparations we can prove Theorem 2.4.

Proof of Theorem 2.4. We choose $s > 1$ such that $s\rho_1 < 1$. We let ϕ_0 be the piecewise linear B-Spline. Since both ϕ_0 and the refinable function ϕ associated with S reproduce constant functions and have compact support, the inequality

$$\|p * \phi_0 - p * \phi\|_\infty \le C_1 \sup\nolimits_{|\mu|=1} \|\nabla^\mu p\|_\infty$$

holds for all bounded input data p with C_1 not depending on p. Furthermore,

$$\|p * \phi_0\|_\infty \le C_2 \|p\|_\infty \quad \text{and} \quad \|p * \phi\|_\infty \le C_3 \|p\|_\infty ,$$

where the constants are the corresponding operator norms. Let C_4 be the constant from the first order proximity condition, and C_5 be the constant from (4.23). We use the symbol σ_1'' for the constant from (4.26). Then we let

$$\sigma'' = \min\Big(\sigma_1'', \frac{\sigma'}{4C_1C_5}, \Big(\frac{\sigma'(1-\rho_1^2 s^2)}{2C_3C_4C_5^2}\Big)^{1/2}, \frac{\sigma}{C_5}\Big). \tag{4.35}$$

We show that, for input data $p \in P_{N'',\sigma''}$, $T^n p$ is defined for all $n \in \mathbb{N}$, and that $T^n p \in P_{N,\sigma}$. Then the assumptions of Lemma 4.3 are met and we can use this lemma to deduce convergence. We use induction on n. As induction hypothesis we assume that for all $k = 0, \ldots, n$, $T^k p$ is well-defined and that it belongs to $P_{N,\sigma}$. Furthermore, we assume that $T^k p$ takes values in $U_{\sigma'}(N'')$. Then $T^{n+1} p$ is defined, and

$$\begin{aligned}
\|T^n p * \phi(M^n \cdot) - T^{n-1} p * \phi(M^{n-1} \cdot)\|_\infty &= \|(T-S) T^{n-1} p * \phi(M^n \cdot)\|_\infty \\
&\le C_3 C_4 \sup_{|\mu|=1} \|\nabla^\mu T^{n-1} p\|_\infty^2 .
\end{aligned}$$

Using Lemma 4.3 and the above estimate we obtain, for $m < n$,

$$\begin{aligned}
&\|T^n p * \phi_0(M^n \cdot) - T^m p * \phi_0(M^m \cdot)\|_\infty \\
&\quad \le \|T^n p * (\phi_0(M^n \cdot) - \phi(M^n \cdot))\|_\infty + \sum_{k=m}^{n-1} \|T^{k+1} p * \phi(M^{k+1} \cdot) - T^k p * \phi(M^k \cdot)\|_\infty \\
&\qquad + \|T^m p * (\phi_0(M^m \cdot) - \phi(M^m \cdot))\|_\infty \\
&\quad \le C_1 \sup_{|\mu|=1} \|\nabla^\mu T^n p\|_\infty + C_3 C_4 \sum_{k=m}^{n-1} \sup_{|\mu|=1} \|\nabla^\mu T^k p\|_\infty^2 + C_1 \sup_{|\mu|=1} \|\nabla^\mu T^m p\|_\infty \\
&\quad \le 2 C_1 C_5 (\rho_1 s)^m \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty + C_3 C_4 C_5^2 \sum_{k=m}^{n-1} (\rho_1 s)^{2k} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 \\
&\quad \le \tfrac{\sigma'}{2} (\rho_1 s)^m + C_3 C_4 C_5^2 (\rho_1 s)^m (1 - \rho_1^2 s^2)^{-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 \le \sigma' (\rho_1 s)^m. \qquad (4.36)
\end{aligned}$$

The last inequality is true because of the choice of σ'' in (4.35) and because, by assumption, $\|\nabla^\mu p\|_\infty \le \sigma''$. If we let $m = 0$ in (4.36), we obtain that $T^{n+1} p$ takes values in $U_{\sigma'}(N'')$. Furthermore, $\sup_{|\mu|=1} \|\nabla^\mu T^{n+1} p\|_\infty \le C_5(\sigma/C_5) = \sigma$. This completes the induction.

A straightforward consequence of (4.36) is that $T^{n+1} p * \phi_0(M^{n+1} \cdot)$ is a Cauchy sequence, which implies the convergence of T for input data p which belong to $P_{M'',\sigma''}$. Furthermore,

$$\|T^n p * (\phi_0(M^n \cdot) - \phi(M^n \cdot))\|_\infty \le C_1 \sup_{|\mu|=1} \|\nabla^\mu T^n p\|_\infty,$$

and the right hand side approaches 0 as $n \to \infty$. This implies that the sequence of uniformly continuous functions $T^n p * \phi(M^n \cdot)$ converges to the limit of T for input p as $n \to \infty$. Hence, we also have that the limit f_p of the nonlinear scheme is uniformly continuous. This completes the proof. □

Our next objective is the proof of our main result on smoothness of nonlinear subdivision schemes in the regular grid case.

Proof of Theorem 2.5. We show that $T^n p * \phi(M^n \cdot)$ is a Cauchy sequence in Lip_γ. Then Theorem 2.4 implies that the limit function of T belongs to Lip_γ. We choose $s > 1$ such that $s^2 \rho_1 < 1$. We let C_1 be the constant of Proposition 4.2 and C_2 be the proximity constant of (2.14), and we denote the constant of (4.24) by L. We use Proposition 4.2 to estimate

$$\begin{aligned} &|T^{n+1}p * \phi(M^{n+1}\cdot) - T^n p * \phi(M^n\cdot)|_{\mathrm{Lip}_\gamma,k} \\ &\qquad = |T^{n+1}p * \phi(M^{n+1}\cdot) - ST^n p * \phi(M^{n+1}\cdot)|_{\mathrm{Lip}_\gamma,k} \\ &\qquad \le C_1 |\lambda_{\max}|^{\gamma n} s^n \sup_{|\mu|=k} \|\nabla^\mu (S-T)T^n p\|_\infty \\ &\qquad \le 2C_1C_2|\lambda_{\max}|^{\gamma n} s^n \Omega_k(T^n p). \end{aligned} \tag{4.37}$$

By (4.24),

$$\Omega_k(T^n p) \le L(\rho_k \rho_1 s)^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2. \tag{4.38}$$

By the definition of the smoothness index $\nu(a, M)$, we have $\rho_k = |\lambda_{\max}|^{-\nu(a,M)}$. Therefore, $\rho_k|\lambda_{\max}|^\gamma < 1$. Using this fact and plugging (4.38) into (4.37) we get

$$|T^{n+1}p * \phi(M^{n+1}\cdot) - T^n p * \phi(M^n\cdot)|_{\mathrm{Lip}_\gamma,k} \le 2C_1C_2L\ r^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2,$$

where $r = \rho_k|\lambda_{\max}|^\gamma s^2 |\lambda_{\min}|^{-1} < 1$. We apply this estimate to obtain

$$\begin{aligned} &\|T^{n+l}p * \phi(M^{n+l}\cdot) - T^n p * \phi(M^n\cdot)\|_{\mathrm{Lip}_\gamma,k} \\ &\le |T^{n+l}p * \phi(M^{n+l}\cdot) - T^n p * \phi(M^n\cdot)|_{\mathrm{Lip}_\gamma,k} + \|T^{n+l}p * \phi(M^{n+l}\cdot) - T^n p * \phi(M^n\cdot)\|_\infty \\ &\le C_1C_2L\ r^n(1-r)^{-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 \cdot + \|T^{n+l}p * \phi(M^{n+l}\cdot) - T^n p * \phi(M^n\cdot)\|_\infty, \end{aligned}$$

where the second term tends to 0 by Theorem 2.4. Therefore, $T^n p * \phi(M^n\cdot)$ is a Cauchy sequence in Lip_γ. This completes the proof. □

Proof of Theorem 2.6. It remains to verify the proximity inequalities. The geometric analogues considered in this corollary are instances of the so-called g-f-analogues intoduced in [26]. Therefore the proximity inequalities for the intrinsic mean analogue (2.10), the log-exp analogue (2.11), and the projection analogue (2.12) follow directly from Theorems 5.8 and 5.9 of [7]. □

Proof of Corollary 2.7. Theorem 2.6 ensures that the mentioned analogues produce limits f_p whose smoothness index $\nu(f_p)$ is at least as high as the smoothness index $\nu(a, M)$ of the linear scheme. Then the smoothness index of the refinable function $\nu(\phi)$ equals the smoothness index $\nu(a, M)$ [9]. The second statement of the corollary follows from the fact that if $S_{a,M}$ produces C^k limits, then the corresponding smoothness index $\nu(a, M)$ is strictly greater than k [10]. □

4.2. Proofs for the case of irregular combinatorics

The purpose of this part is to prove Theorem 3.4 and Theorem 3.5. Our first task is to establish contractivity of a nonlinear scheme T which is in proximity to a linear scheme S (as in Section 3.3) near the singularity. We need the following lemma which is concerned with the contractivity of the subdivision matrix of S.

Lemma 4.4. *Let $A: \mathbb{R}^m \to \mathbb{R}^m$ be a matrix with dominant single eigenvalue 1 for the eigenvector $v_1 = (1,\dots,1)^T$. We denote a subdominant eigenvalue of A by λ. We let $\Delta'(b) = \sup_{1\le k,j\le m} |b_k - b_j|$ for $b \in \mathbb{R}^m$. Then for every $s > 1$ there is $C > 0$ such that,*

$$\Delta'(A^l b) \le C(|\lambda|s)^l \Delta'(b), \qquad \textit{for all } l \in \mathbb{N}, \textit{ and all } b \in \mathbb{R}^m. \tag{4.39}$$

This is Lemma 2.3 of [25]. Together with the proximity condition (3.17) we use it to establish the next lemma which involves differences of subdivided data near the extraordinary point. We employ the following notation: We consider data p_n defined on V_n for some level n. For a subset $B \subset V_n$, we let

$$\mathcal{D}_B(p_n) := \sup\{|p_n(x) - p_n(y)| : \ x, y \in B, x \text{ and } y \text{ are face-neighbors }\},$$

If $B = V_n$ we drop the lower index.

Lemma 4.5. *Assume that a linear scheme S as defined in Section 3.3 and the scheme T fulfill the local proximity condition* (3.17) *w.r.t σ-dense input $P_{N,\sigma}$. Then for $s > 1$ there is a constant $C > 0$ and $\sigma'' > 0$ such that the following is true: If the input data p_0 belongs to $P_{N,\sigma''}$, if iterated subdivision for input p_0 is defined, and if $T_{l-1,0}p_0$ stays within $P_{N,\sigma}$ for all $l \le n$, then*

$$\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(T_{n-1,0}p_0) \le C(\lambda s)^n \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0), \tag{4.40}$$

where λ is the subdominant eigenvalue of the subdivision matrix of S.

Proof. We start by rephrasing (4.39). For any $s > 1$ there is a constant $C_L \ge 1$ such that for all levels n and all data p_n on level n the following is true: The linear scheme is contractive for data on the control sets of the inner areas D'_n (defined by (3.16)) in the following sense

$$\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(S_{n-1,0}p_0) \le C_L(\lambda s)^n \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0). \tag{4.41}$$

To see this, we consider the definition of the subdivision matrix A of the scheme S in Section 3.3. The subdivision matrix A maps data on $\mathrm{ctrl}^0(D'_0)$ to subdivided data on $\mathrm{ctrl}^1(D'_1)$. Therefore, A^n maps data on $\mathrm{ctrl}^0(D'_0)$ to n-times subdivided data on $\mathrm{ctrl}^n(D'_n)$. In this interpretation, (4.39) estimates differences of $\dim(A)$ many subdivided data items by $\dim(A)$ many input data items. Therefore, application of the triangle inequality and enlarging the constant C of (4.39) yields (4.41).

We also rewrite the local proximity condition (3.17): There is a constant C_P such that for σ-dense input $p_n \in P_{N,\sigma}$ on some data level n,

$$\|S_n p_n(w) - T_n p_n(w)\|_\infty \leq C_P(\mathcal{D}_{\mathrm{supp}(\alpha_{\cdot,w})}(p_n))^2, \tag{4.42}$$

where $\mathrm{supp}(\alpha_{\cdot,w})$ denotes the set of vertices on level n which contribute to the calculation of $S_n p(w)$. Equation (4.42) is derived from (3.17) by using the triangle inequality and and the fact that $(a+b)^2 \leq 2(a^2+b^2)$ for $a, b \in \mathbb{R}$. Hereby the constant C of (3.17) is enlarged.

By the locality of the proximity condition, control sets for S are control sets for T, too. Thus an immediate consequence of (4.42) is that

$$\|(S_n p_n - T_n p_n)|_{\mathrm{ctrl}^{n+1}(D'_{n+1})}\|_\infty \leq C_P(\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(p_n))^2. \tag{4.43}$$

With these preparations we define the 'denseness'-bound σ'' by

$$\sigma'' = \frac{(1-\lambda s)\lambda s}{8 C_P C_L^2}. \tag{4.44}$$

For data p_0 meeting the requirements of the lemma we show that

$$\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(T_{n-1,0}p_0) \leq 2C_L(\lambda s)^n \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0), \tag{4.45}$$

using induction on n. This implies (4.40) with $C = 2C_L$. We start with $n = 1$ and estimate

$$\begin{aligned}
\mathcal{D}_{\mathrm{ctrl}^1(D'_1)}(T_0 p_0) &\leq \mathcal{D}_{\mathrm{ctrl}^1(D'_1)}(T_0 p_0 - S_0 p_0) + \mathcal{D}_{\mathrm{ctrl}^1(D'_1)}(S_0 p_0) \\
&\leq 2\|T_0 p_0 - S_0 p_0)_{\mathrm{ctrl}^1(D'_1)}\|_\infty + \mathcal{D}_{\mathrm{ctrl}^1(D'_1)}(S_0 p_0) \\
&\leq 2C_P(\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0))^2 + C_L(\lambda s)\mathcal{D}_{V_0}(p_0) \\
&\leq C_L(2C_P \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0) + \lambda s)\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0) \\
&\leq 2C_L(\lambda s)\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0).
\end{aligned}$$

The second inequality estimates differences by twice the sup-norm of data. For the third inequality we used proximity in the form of (4.43) and the contractivity of the linear scheme near the singularity in the form of (4.41). For the fourth inequality notice that $C_L \geq 1$. The last inequality is a consequence of our choice of σ'' in (4.44). As induction hypothesis we assume that (4.45) is true for all $l < n$. We now show (4.45) by estimating

$$\begin{aligned}
&\mathcal{D}_{\mathrm{ctrl}^n(D_n')}(T_{n-1,0}p_0) \\
&\qquad \le \sum\nolimits_{l=1}^{n} \mathcal{D}_{\mathrm{ctrl}^n(D_n')}(S_{n-1,l}T_{l-1,0}p_0 - S_{n-1,l-1}T_{l-2,0}p_0) + \mathcal{D}_{\mathrm{ctrl}^n(D_n')}(S_{n-1,0}p_0) \\
&\qquad \le \sum\nolimits_{l=1}^{n} C_L(\lambda s)^{n-l}\mathcal{D}_{\mathrm{ctrl}^l(D_l')}(T_{l-1,0}p_0 - S_{l-1}T_{l-2,0}p_0) + \mathcal{D}_{\mathrm{ctrl}^n(D_n')}(S_{n-1,0}p_0) \\
&\qquad \le \sum\nolimits_{l=1}^{n} 2C_L(\lambda s)^{n-l}C_P(\mathcal{D}_{\mathrm{ctrl}^{l-1}(D_{l-1}')}(T_{l-2,0}p_0))^2 + C_L(\lambda s)^k\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0).
\end{aligned}$$

For the second inequality we used the contractivity of S near the singularity in the sense of (4.41). For the third inequality we estimated differences by twice the sup-norm and then applied the proximity inequality (4.43). We use the induction hypothesis and obtain

$$\begin{aligned}
&\mathcal{D}_{\mathrm{ctrl}^n(D_n')}(T_{n-1,0}p_0) \\
&\quad \le \sum\nolimits_{l=1}^{n} 8C_LC_P(\lambda s)^{n-l}C_L^2(\lambda s)^{2(l-1)}(\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0))^2 + C_L(\lambda s)^n\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0) \\
&\quad \le C_L\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)\left[C_L^2\sum\nolimits_{l=1}^{k} 8C_P(\lambda s)^{n+l-2}\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0) + (\lambda s)^n\right] \\
&\quad \le C_L(\lambda s)^n\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)\left[\frac{8C_PC_L^2}{(1-\lambda s)\lambda s}\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0) + 1\right] \\
&\quad \le 2C_L(\lambda s)^n\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0).
\end{aligned}$$

For the first inequality we use the contractivity of T which is the induction hypothesis. The last inequality is true by our choice of σ''. This completes the induction. □

We have collected all information necessary to show convergence. Below, Theorem 4.6 gives the precise version of Theorem 3.4. The formulation is rather technical, which is mainly due to the fact that nonlinear schemes are in general not globally defined. We therefore have to guarantee the well-definedness of the data during the subdivision process.

Theorem 4.6. *Let S and T fulfill a local proximity condition w.r.t. some $P_{N,\sigma}$. Assume that T_np_n takes its values in some set N', where $N \subset N' \subset \mathbb{R}^d$, for all input data p_n in $P_{N,\sigma}$ on all level n. Assume further that there is $N'' \subset N$ and $\sigma' > 0$ such that the σ'-neighborhood $U_{\sigma'}(N'')$ obeys $U_{\sigma'}(N'') \cap N' \subset N$. Then there is $\sigma'' > 0$ such that T converges for $p \in P_{N'',\sigma''}$, and*

$$S_{\infty,i+1}T_{i,0}p \to T_{\infty,0}p \quad \text{as} \quad i \to \infty. \tag{4.46}$$

This convergence is in the sense of the sup *norm.*

Proof. We split the proof of this statement into several parts. In part (1) we obtain the contractivity of the nonlinear scheme T, where we assume that $T_{n,0}p_0$ is defined for all n and certain input data p_0. In part (2) we define interpolation operators which extend the discrete data on different levels to continuous functions and derive some properties. In part (3) we define the constant σ'' and explain our choice of σ''. In part (4) we apply the interpolation operators from part (2) to show that that subdivision by T is well defined for σ''-dense data p_0 in $P_{N'',\sigma''}$, thus justifying the assumption of (1). Furthermore, we use the proximity of S and T and the contractivity of T to derive the convergence of T for data in $P_{N'',\sigma''}$. In part (5), we use part (4) and the interpolation operator from part (2) to show (4.46).

(1) In this part we obtain contractivity of T. We denote the subdominant eigenvalue of the subdivision matrix of the linear scheme S by the symbol λ, and we let M be the dilation matrix corresponding to S. We choose $s > 1$ such that

$$\gamma := s \max(|\lambda|, 1/\sqrt{\det M}) < 1.$$

We show that there is σ_1'' and $C_1 \geq 1$ such that the following is true: If input data p_0 on level 0 belongs to $P_{N,\sigma''}$, if iterated subdivision for input p_0 is defined, and if $p_l = T_{l-1,0}p_0$ stays within $P_{N,\sigma}$ for all $l \leq n$, then

$$\mathcal{D}(T_{n-1,l}p_l) \leq C_1 \gamma^{n-l} \mathcal{D}(p_l). \tag{4.47}$$

This is a consequence of the corresponding statement near the singularity which is formulated in Lemma 4.5 and the corresponding statement for the regular mesh case which is Lemma 4.3. The constant C_1 is the product of the corresponding constants of Lemma 4.3 and Lemma 4.5, and σ_1'' is obtained as follows: We apply Lemma 4.5 for the denseness bound σ used in the statement of the theorem. We obtain a constant $\sigma''_{\text{Lemma 4.5}}$. Then we apply Lemma 4.3 for this constant, i.e., we replace the σ in Lemma 4.3 by $\sigma''_{\text{Lemma 4.5}}$. The resulting denseness bound is denoted by σ_1''.

In order to conclude (4.47), one has to show that 'no interaction takes place between the neighborhood of the singularity and the regular part': To that end we split the domain D into the inner area D_n' (defined by (3.16)), the rings D_i, $i = 0, \ldots, n-1$ and the 'outer' ring

$$D_{-1} = D \setminus D'.$$

The union of the corresponding n-th level control sets equals V_n and control sets of neighboring items of the splitting overlap (recall that control sets were defined w.r.t. the linear scheme S which are also control sets w.r.t. T by the local proximity condition). We consider (4.47) separately on the items of the splitting: The control set of the outer ring D_{-1} intersected with each sector has regular combinatorics on all data levels. Therefore the validity of (4.47) on $\mathrm{ctrl}^n(D_{-1})$ is a consequence of Lemma 4.3. On D'_n, (4.47) is a direct consequence of Lemma 4.5 applied to $\mathrm{ctrl}^n(D'_n)$. We consider the rings D_i : For each segment D_i^j of the i-th ring we consider its n-th level control set and get

$$\begin{aligned}\mathcal{D}_{\mathrm{ctrl}^n(D_i^j)}(T_{n-1,0}p_0) &\le C_{\text{Lemma 4.3}}(s\det M)^{(n-i)/2}\mathcal{D}_{\mathrm{ctrl}^i(D_i^j)}(T_{i-1,0}p_0)\\ &\le C_{\text{Lemma 4.3}}C_{\text{Lemma 4.5}}(s\det M)^{(i-n)/2}(s\lambda)^i\mathcal{D}_{\mathrm{ctrl}^0(D_0)}(p_0) \le C_1\gamma^n\mathcal{D}(p_0).\end{aligned}$$

Altogether, this shows (4.47) and completes part (1).

(2) The convergence of subdivision with T is quite intricate. This mostly comes from the fact that the well-definedness of iterated application of T has to be guaranteed. For that we need interpolation operators I_i which map data on level i to a uniformly continuous function on the domain D. The domain D is perfectly suited to smoothness analysis across sector boundaries (not near the central point). However, in this part we are only concerned with convergence and we use a homeomorphism $E : D \to \mathbb{R}^2$ to reparametrize data on each level, and to reparametrize limit functions. E maps entire D to the plane by first squeezing the j-th sector into a sector of opening angle $2\pi/k$ with a shear transformation and then rotating it by an angle of $2\pi j/k$. It is straightforward to see that there are constants c_1, c_2 such that for $x, y \in D$,

$$c_1 \operatorname{dist}(x, y) \le \operatorname{dist}(E(x), E(y)) \le c_2 \operatorname{dist}(x, y).$$

This implies that convergence of a scheme is invariant under reparametrization by means of E.

The points $E(V_i)$ are still associated with a k-regular combinatorics. By connecting points in $E(V_i)$ with straight lines according to the combinatorics we get a realization of its edges and faces in $\mathbb{R}^2$. For defining the interpolation operator $\bar{I}_i$ which maps data on $E(V_i)$ to a function on $\mathbb{R}^2$ we split each face into triangles, each of them determined by the face's barycenter and an edge. We get data for the barycenter by the barycenter of the data on the neighboring vertices. Then we use linear interpolation on the triangles. For x, y in a face and data p_n defined on $E(V_n)$, we obviously have

$$\sup_{x,y \text{ belong to the same face}} \|\bar{I}_n p_n(x) - \bar{I}_n p_n(y)\|_{\mathbb{R}^d} \le \mathcal{D}(p_n). \tag{4.48}$$

Furthermore the infimum d' of distances of neighboring vertices in $E(V_i)$ satisfies

$$c_3(\det M)^{-i/2} \le d' \le \{\operatorname{diam} F : F \text{ is a face on level } i\} \le c_4(\det M)^{-i/2}, \tag{4.49}$$

where the constants c_3, c_4 are independent of the the level i. In addition, there is a constant R for all levels i such that the value

$$S_i p_i(v) \text{ is an affine average of } \{p_i(w) : \ w \in B(v, (\det M)^{-i/2}R)\}. \tag{4.50}$$

Here the considered points w are elements of $E(V_i)$, and $B(x,r)$ is the open ball with radius r around x.

Interpolation operators I_i mapping data on V_i to functions on D are obtained from the operators $\bar{I}_i$ by reversing the reparametrization E.

The interpolation operators I_i have the following properties: There are constants $C_B, C_I > 0$, which depend neither on i nor on bounded data p_i on level i, such that

$$\|I_{i+1}S_i p_i - I_i p_i\| \leq C_B \mathcal{D}(p_i), \tag{4.51}$$

$$\|S_{\infty,i}p_i|_{V_i} - p_l\| \leq \|S_{\infty,i}p_i - I_i p_i\| \leq C_I \mathcal{D}(p_i). \tag{4.52}$$

When showing (4.51) and (4.52) we may replace I_i by $\bar{I}_i$, and we may reparametrize both data and limit functions using the map E. This is justified, since a reparametrization does not effect the statements. We begin with (4.51). For arbitrary $x \in \mathbb{R}^2$ we choose faces F_i and F_{i+1} containing x on levels i and $i+1$, respectively. We consider vertices v_i of F_i and v_{i+1} of F_{i+1} and estimate

$$\begin{aligned}
&\|\bar{I}_{i+1}S_i p_i(x) - \bar{I}_i p_i(x)\| \\
&\leq \|\bar{I}_{i+1}S_i p_i(x) - \bar{I}_{i+1}S_i p_i(v_{i+1})\| + \|S_i p_i(v_{i+1}) - p_i(v_i)\| + \|\bar{I}_i p_i(v_i) - \bar{I}_i p_i(x)\| \\
&\leq \mathcal{D}(S_i p_i) + \mathcal{D}(p_i) + \|S_i p_i(v_{i+1}) - p_i(v_i)\|.
\end{aligned}$$

In order to estimate the last summand on the right hand side, note that by (4.50) the value $S_i p_i(v_{i+1})$ is uniquely determined by $p_i|_{E(V_i)\cap B(v_{i+1},(\det M)^{-i/2}R)}$. With the constant c_4 of (4.49) it follows that $\operatorname{dist}(v_i, v_{i+1}) \leq 2c_4(\det M)^{-i/2}$. Consequently, $\max\{\operatorname{dist}(v_i, y) : \ y \in E(V_i) \cap B(v_{i+1}, 2^{-i}R)\} \leq 2c_4(\det M)^{-i/2} + (\det M)^{-i/2}R$. The left hand inequality in (4.49) now implies that the number of faces on level i which are not disjoint to the ball $B(v_i, (2c_4+R)(\det M)^{-i/2})$ is bounded by some integer D which is independent of the level i and v_i. With $B^* := B(v_{i+1}, (\det M)^{-i/2}R)$, we can write $S_i p_i(v_{i+1}) = \sum_{q\in E(V_i)\cap B^*} \alpha_q p_i(q)$ with $\sum_{q\in E(V_i)\cap B^*} \alpha_q = 1$ and $\sum_{q\in E(V_i)\cap B^*} |\alpha_q| \leq \|S_i\|$. We obtain

$$\begin{aligned}
\|S_i p_i(v_{i+1}) - p_i(v_i)\| &= \left\| \sum\nolimits_{q\in E(V_i)\cap B^*} \alpha_q (p_i(q) - p_i(v_i)) \right\| \\
&\leq \sum\nolimits_{q\in E(V_i)\cap B^*} |\alpha_q| \cdot \max\nolimits_{q\in E(V_i)\cap B^*} \|p_i(q) - p_i(v_i)\| \leq \|S_i\| D \mathcal{D}(p_i).
\end{aligned}$$

Altogether, it follows that

$$\|\bar{I}_{i+1}S_i p_i - \bar{I}_i p_i\| \leq \mathcal{D}(S_i p_i) + (\|S_i\|D + 1)\mathcal{D}(p_i). \tag{4.53}$$

This implies (4.51), since $\|S_i\|$ is uniformly bounded in i.

We show (4.52) for the interpolation operators $\bar{I}_i$. Equipped with (4.53), we estimate, for $n > i$,

$$\begin{aligned}\|\bar{I}_{n+1}S_{n,i}p_i - \bar{I}_n S_{n-1,i}p_i\|_\infty \leq & \mathcal{D}S_{n,i}p_i + (\|S_n\|D + 1)\mathcal{D}(S_{n-1,i}p_i) \\ \leq & C_1\gamma^{n-i}(\|S_n\|D + 2)\mathcal{D}(p_i),\end{aligned}$$

where we used the contractivity of S which follows, for example, from part (1), since S can be seen as a scheme in proximity to S. For $n'' \geq n' \geq n \geq i$ we make use of the geometric series and get

$$\|\bar{I}_{n''+1}S_{n'',i}p_i - \bar{I}_{n'}S_{n'-1,i}p_i\|_\infty \leq C(\sup_{n\in\mathbb{N}_0} \|S_n\|D + 2)\gamma^{n-i}\frac{1}{1-\gamma}\mathcal{D}(p_i). \tag{4.54}$$

Thus $\{\bar{I}_n S_{n-1,i}p_i\}_{n>i}$ is a Cauchy sequence in the space of bounded continuous functions. Since these functions are uniformly continuous, so is the limit, called f for the moment. Now, $\|f|_{E(V_n)} - S_{n-1,i}p_i\|_\infty \leq \|f - \bar{I}_n S_{n-1,i}p_i\| \to 0$ for $n \to \infty$. Thus f equals $S_{\infty,i}p_i$. Letting $n' = i$ in (4.54) yields the estimate

$$\|f - \bar{I}_i p_i\| = \lim_{n''\to\infty} \|\bar{I}_{n''+1}S_{n'',i}p_i - \bar{I}_i p_i\| \leq \frac{1}{1-\gamma}(\sup_{n\in\mathbb{N}_0} \|S_n\|D + 2)\mathcal{D}(p_i).$$

This implies (4.52).

(3) We define the constant σ'' which guarantees convergence by

$$\sigma'' = \min\Big(\sigma''_1, \frac{\sigma}{C_1}, \frac{1-\gamma}{2C_B C_1}\sigma', \Big(\frac{1-\gamma^2}{2C_P C_1^2}\sigma'\Big)^{\frac{1}{2}}\Big). \tag{4.55}$$

The constant C_B is given by (4.51), and the symbol C_P denotes the proximity constant as used in (4.43). We take C_1, σ''_1 and the contractivity factor γ from part (1). For σ''_1-dense input data p_0, contractivity of T in the sense of (4.47) is guaranteed whenever iterated subdivision for input p_0 is defined, and $T_{l-1,0}p_0$ stays within $P_{N,\sigma}$. The choice of the other items in (4.55) guarantees these two properties as shown in part (4). The second item is important in the estimates (4.56) and (4.58). The last two items are important in the estimates (4.57) and (4.59).

(4) We apply the interpolation operators from part (2) to show that subdivision with T is well defined for σ''-dense data p_0 in $P_{N'',\sigma''}$ and that $T_{i,0}p_0$ stays within $P_{N,\sigma}$ for all i. We use induction on the subdivision level i. We consider input data $p_0 \in P_{N'',\sigma''}$. Since $\mathcal{D}(p_0) < \sigma'' < \sigma$, subdivision by T for input p_0 is defined. From (4.47) we get that

$$\mathcal{D}(T_0 p_0) \leq C_1\gamma\mathcal{D}(p_0) \leq C_1\sigma'' \leq \sigma. \tag{4.56}$$

The last inequality is a consequence of the choice of σ''.

Now we use the interpolation operators from part (2) and get

$$\begin{aligned}\|I_1T_0p_0 - I_0p_0\| &\le \|I_1T_0p_0 - I_1S_0p_0\| + \|I_1S_0p_0 - I_0p_0\| \\ &\le \|T_0p_0 - S_0p_0\| + C_B\mathcal{D}(p_0) \\ &\le C_P\mathcal{D}(p_0)^2 + C_B\mathcal{D}(p_0) \le \tfrac{\sigma'}{2} + \tfrac{\sigma'}{2}. \end{aligned} \tag{4.57}$$

Here we used (4.51) for the second inequality and the proximity condition (4.43) for the third inequality. The last inequality is a consequence of our choice of σ''. From the assumptions we made it follows that T_0p_0 takes its values in N. Combining this fact with (4.56), we get that $T_0p_0 \in P_{N,\sigma}$ and thus T_0p_0 is in the domain of T_1. This serves as the induction base (i=0).

We use as an induction hypothesis that $T_{n-1,0}p_0$ is well-defined, that $T_{n-1,0}p_0$ takes its values in M, and that $T_{n-1,0}p_0$ is in the domain of T_n, for $n = 1, \dots, i$.

From (4.47) we get

$$\mathcal{D}(T_{i,0}p_0) \le C_1\gamma^{i+1}\mathcal{D}(p_0) \le C_1\sigma'' \le \sigma. \tag{4.58}$$

The last inequality is a consequence of the choice of σ''.

Now we use the interpolation operators from part (2) and get

$$\begin{aligned}&\|I_{i+1}T_{i,0}p_0 - I_0p_0\| \\ &\le \sum_{n=0}^{i} \|I_{n+1}T_{n,0}p_0 - I_{n+1}S_nT_{n-1,0}p_0\| + \|I_{n+1}S_nT_{n-1,0}p_0 - I_nT_{n-1,0}p_0\| \\ &\le C_P\sum_{n=0}^{i}\mathcal{D}(T_{n-1,0}p_0)^2 + C_B\sum_{n=0}^{i}\mathcal{D}(T_{n-1,0}p_0) \\ &\le C_PC_1^2\left(\sum_{n=0}^{\infty}\gamma^{2n}\right)\mathcal{D}(p_0)^2 + C_BC_1\sum_{n=0}^{\infty}\gamma^n\mathcal{D}(p_0) \\ &\le \frac{C_PC_1^2}{1-\gamma^2}\mathcal{D}(p_0)^2 + \frac{C_BC_1}{1-\gamma}\mathcal{D}(p_0) \le \tfrac{\sigma'}{2} + \tfrac{\sigma'}{2}. \end{aligned} \tag{4.59}$$

Here we used (4.51) and the proximity condition (4.43) for the second inequality. The last inequality is a consequence of our choice of σ''. From our assumptions it follows that $T_{i,0}p_0$ takes its values in N. Combining this fact with (4.58) we get that $T_{i,0}p_0 \in P_{N,\sigma}$ and thus $T_{i,0}p_0$ is in the domain of T_{i+1} which means that $T_{i+1,0}p_0$ is well-defined. This completes the induction.

As a consequence, for σ''-dense input in $P_{N'',\sigma''}$, $T_{i,0}p_0$ exists for all i and T is contractive for such input in the sense of (4.47). Toward convergence, we choose $i'' \ge i' \ge i$ and estimate

$$\begin{aligned}\|I_{i''+1}T_{i'',0}p_0 - I_{i'+1}T_{i',0}p_0\| &\le \frac{C_PC_1^2}{1-\gamma^2}\mathcal{D}(T_{i'-1,0}p_0)^2 + \frac{C_BC_1}{1-\gamma}\mathcal{D}(T_{i'-1,0}p_0) \\ &\le \frac{C_PC_1^4}{1-\gamma^2}\gamma^{2i}\mathcal{D}(p_0)^2 + \frac{C_BC_1^2}{1-\gamma}\gamma^i\mathcal{D}(p_0). \end{aligned}$$

Since the right hand side approaches 0 as $i \to \infty$, the sequence $\{I_i T_{i-1,0} p_0\}_{i \in \mathbb{N}}$ is a Cauchy sequence in $C(D, \mathbb{R}^d)$ and therefore convergent. Each sequence member is uniformly continuous, which implies the same for the limit. Thus T converges for input in $P_{N'',\sigma''}$.

(5) It remains to show (4.46). We consider $\varepsilon > 0$, and choose the index L large enough such that for all indices $i \geq L$, $\|T_{\infty,0} p_0 - I_i T_{i-1,0} p_0\| < \frac{\varepsilon}{2}$. With (4.52) we estimate, for $i \geq L$,

$$\|S_{\infty,i} T_{i-1,0} p_0 - I_i T_{i-1,0} p_0\| \leq C_I \mathcal{D}(T_{i-1,0} p_0) \leq C_I C_1 \gamma^i \mathcal{D}(p_0).$$

Now we choose $L_0 > L$ such that $C_I C_1 \gamma^{L_0} < \frac{\varepsilon}{2}$. Then for all $i \geq L_0$, $\|T_{\infty,0} p_0 - S_{\infty,i} T_{i-1,0} p_0\| < \varepsilon$. This proves (4.46). □

Our next task is to prove Theorem 3.5 which is a smoothness statement. For that we need the following two lemmas concerning the characteristic parametrization of limit functions. We refer to [18] for a detailed exposition of the characteristic parametrization.

Lemma 4.7. *Let λ be the subdominant eigenvalue of the subdivision matrix A of a linear subdivision scheme as defined in Section 3.3 (which has the single dominant eigenvalue 1). If we choose the ring index n_0 sufficiently large, we get a constant $C > 0$ such that, for all $n \geq n_0$ and each C^1 function $f : D_n \to \mathbb{R}^d$,*

$$\|f \circ \chi^{-1}\|_{C^1(\chi(D_n),\mathbb{R}^d)} \leq C|\lambda|^{-n} (\det M)^{-n/2} \|f\|_{C^1(D_n,\mathbb{R}^d)} \tag{4.60}$$

(M is the dilation matrix, D_n is the n-th ring). The constant C does not depend on the ring index $n \geq n_0$.

Proof. By our assumptions on the linear scheme S, its characteristic map χ is 1-1 in a neighborhood of the point 0. So we find an index n_0, such that χ is 1-1 on D'_{n_0}. In the following we assume that n_0 is chosen such that this requirement is fulfilled.

Our argument is based on the following fact which we verify only at the end of the proof: There is a ring index n_0 and a constant $C > 0$ such that the differential of the characteristic map χ obeys

$$\sup_{x \in D_n} \|d_x \chi_n(v)\| \geq C|\lambda|^n (\det M)^{n/2} \|v\|, \tag{4.61}$$

where C is independent of the ring index $n \geq n_0$ and the point $x \in D_n$. We use the Euclidean norm for the tangent vectors v; $\|d_x \chi_n^{-1}\|$ is the induced operator norm. In other words, (4.61) states that differentials are lower bounded, uniformly for all $x \in D_n$, with constant C independent of the ring. If (4.61) is proved, we can apply the inverse function theorem to obtain a constant $C > 0$ such that

$$\sup_{y \in \chi(D_n)} \|d_y \chi_n^{-1}\| \leq C|\lambda|^{-n}(\det M)^{-n/2}, \tag{4.62}$$

where C is independent of the ring index $n \geq n_0$. Using the submultiplicativity of operator norms we get

$$\|d_y f \circ \chi^{-1}\| \leq \|d_{\chi^{-1}(y)} f\| \cdot \|d_y \chi_n^{-1}\| \leq C|\lambda|^{-n}(\det M)^{-n/2}\|f\|_{C^1(D_n,\mathbb{R}^d)}.$$

This implies (4.60), since sup-norms of functions do not change under reparametrization.

To show (4.61) we need some preparations. We consider a Jordan block of the subdivision matrix A corresponding to a subdominant eigenvalue λ. We denote its multiplicity by m and order the Jordan vectors w_i, such that w_0 is the eigenvector. For the Jordan vector with the highest multiplicity, we have the expression

$$A^n w_{m-1} = \sum_{i=0}^{m-1} \binom{n}{i} \lambda^{n-i} w_{m-i-1}. \tag{4.63}$$

Since $\binom{n}{i}$ grows as n^i as $n \to \infty$, the dominating term in this expression is given by $\binom{n}{m-1}\lambda^{n-m+1}w_0$. If the subdominant eigenvalues of A are complex conjugate numbers, we use the vectors w_i to define new vectors v_i where each component consists of the tuple of real number consisting of the real and the imaginary part of the corresponding component of w_i. If the subdominant eigenvalues of A are real and equal, we use vectors w_i as above and second set of vectors $\bar{w}_i$ corresponding to the second subdominant Jordan block with the same ordering as above. We define new vectors v_i where each component consists of the tuple of real numbers consisting of the corresponding components of w_i and $\bar{w}_i$, respectively.

Then the characteristic χ is the limit of subdivision for the input data stored in the vector v_{m-1}. We write χ_n for the restriction of χ to the ring D_n. We define $\xi_n : D_0 \to \mathbb{R}^2$ by

$$\chi_n = \xi_n \circ (G^n)^{-1}. \tag{4.64}$$

Then ξ_n is the limit function on D_0 of linear (regular mesh) subdivision for 0-th level input data obtained from $A^n v_{m-1}$.

We let $\psi : D_0 \to \mathbb{R}^2$ be the limit function for input data on level 0 obtained from v_0 (which plays an special role) and let $f_i : D_0 \to \mathbb{R}^2$ be the limit functions for the other v_i. All these limits are C^1 on D_0, since they were obtained by regular mesh subdivision. Furthermore, the finiteness of the control sets $\mathrm{ctrl}^0(D_0)$ yields

$$\|S_{\infty,0} p_0\|_{C^1(D_0)} \leq C\|p_0\|_\infty,$$

for arbitrary input data p_0 on $\mathrm{ctrl}^0(D_0)$. Knowing this and the fact that $A^n v_{m-1}$ is dominated by $\binom{n}{m-1}\lambda^{n-m+1}v_0$ for $n \to \infty$, which is a consequence of (4.63), we see that the sequence of mappings

$$\binom{n}{m-1}^{-1}\lambda^{m-n-1}\xi_n \to \psi \quad \text{in} \;\; C^1(D_0),$$

as n tends to ∞. This implies that ψ is regular, since we assumed that ξ_n (which is a reparametrization and restriction of the characteristic map) is regular for sufficiently large n. This fact allows us to estimate the Jacobian of ξ_n from below as follows: We start out by using the inverse triangle inequality

$$\|d_x\xi_n(v)\| = \|\binom{n}{m-1}\lambda^{n-m+1}d_x\psi(v) + \sum\nolimits_{i=0}^{m-2}\binom{n}{i}\lambda^{n-i}d_x f_{m-i-1}(v)\| \tag{4.65}$$

$$\geq \binom{n}{m-1}|\lambda|^{n-m+1}\|d_x\psi(v)\| - \sum\nolimits_{i=0}^{m-2}\binom{n}{i}|\lambda|^{n-i}\|d_x f_{m-i-1}\|\|v\|. \tag{4.66}$$

We use that $\binom{n}{i}$ grows as n^i as $n \to \infty$ to estimate the binomial coefficients. Due to the compactness of D_0 we find a constant $C > 0$ such that for all points $x \in D_0$ and all functions f_i the differentials obey $\|d_x f_i\| \leq C$. Since ψ is regular we get a lower constant $c > 0$ such that, for all $x \in D_0$, $\|d_x\psi(v)\| \geq c\|v\|$. Making the constant c smaller (which comes form estimating the binomial coefficients and multiplying with λ^{m-1}) these estimates help us to get

$$\|d_x\xi_n(v)\| \geq c\, n^{m-1}|\lambda|^n\|v\| - C\, \sum\nolimits_{i=1}^{m-1} n^i|\lambda|^n\|v\|. \tag{4.67}$$

If we now choose n_0 large enough, there is a constant $c > 0$ which does not depend on the index $n > n_0$ such that

$$\|d_x\xi_n(v)\| \geq cn^{m-1}|\lambda|^n\|v\|. \tag{4.68}$$

With (4.64) we get

$$\|d_x\chi_n(v)\| \geq \min_z \|d_z\xi_n(v)\| \det M^{n/2} \geq c|\lambda|^n \det M^{n/2}\|v\|.$$

This proves (4.61). □

Lemma 4.8. *Let p_n be input data on the control set $\mathrm{ctrl}^n(D_n')$ of the inner area D_n' for data level n. Then for large enough n_0, and $s > 1$, there is a constant $C > 0$, which does not depend on the level $n \geq n_0$ and data p_n, such that*

$$\|S_{\infty,n}p_n \circ \chi^{-1}\|_{C^1(\chi(D_n'),\mathbb{R}^d)} \leq C|\lambda|^{-n}s^n\|p_n|_{\mathrm{ctrl}^n(D_n')}\|_\infty. \tag{4.69}$$

Proof. We use the notation of the proof of Lemma 4.7 and choose the integer n_0 so large that χ is regular and injective on $D'_{n_0} \setminus \{0\}$ and such that Lemma 4.7 works. Over the characteristic parametrization, the subdivision scheme S produces C^1 limit functions. As in Lemma 4.7, from the finiteness of the control set $\mathrm{ctrl}^{n_0}(D'_{n_0})$ we conclude that the differential of limit functions w.r.t. the characteristic parametrization can be estimated by

$$\sup_{x \in \chi(D'_{n_0})} \|d_x(S_{\infty,n_0} p_{n_0} \circ \chi^{-1})\| \leq C \|p_{n_0}|_{\mathrm{ctrl}^{n_0}(D'_{n_0})}\|_\infty, \tag{4.70}$$

where the constant C is independent of the n_0-th level input data p_{n_0} given on $\mathrm{ctrl}^{n_0}(D'_{n_0})$. In order to derive (4.69) from (4.70) we consider input p_n on level $n > n_0$, given on the control sets $\mathrm{ctrl}^n(D'_n)$ of the inner area D'_n. Reparametrizing this discrete data with the help of the similarity transform G, i.e., applying G^{n_0-n}, yields data $\bar{p}_{n_0}$ on level n_0. The limit function $S_{\infty,n}p_n$(over D'_n) equals $S_{\infty,n_0}\bar{p}_{n_0} \circ G^{n-n_0}$. Our objective is to get the estimate

$$\sup_{x \in \chi(D'_n)} \|d_x(S_{\infty,n} p_n \circ \chi^{-1})\| \leq C|\lambda|^{-n} s^n \sup_{x \in \chi(D'_{n_0})} \|d_x(S_{\infty,n_0} \bar{p}_{n_0} \circ \chi^{-1})\| \tag{4.71}$$

with the constant C not depending on the level $n > n_0$. If this estimate is established, then (4.69) is a direct consequence of (4.70) if we keep in mind that a reparametrization of any function does not change its sup-norm. To show (4.71), we split D'_n and D'_{n_0} into rings and show (4.71) on the rings. More precisely, we show, letting $r = n - n_0$, that

$$\sup_{x \in \chi(D_{l+r})} \|d_x(S_{\infty,n} p_n \circ \chi^{-1})\| \leq C|\lambda|^{-r} s^r \sup_{x \in \chi(D_l)} \|d_x(S_{\infty,n_0} \bar{p}_{n_0} \circ \chi^{-1})\| \tag{4.72}$$

with the constant C not depending on the $l > n_0$ and $r > 0$. Although the exponents of γ and s in (4.71) and (4.72) differ by n_0 this does not affect the estimate since the resulting constant $\gamma^{n_0} s^{n_0}$ is independent of m and r or n, respectively. Although (4.72) does not consider the central point 0, it nevertheless implies (4.71), since we know that both the function $S_{\infty,n}p_n \circ \chi^{-1}$ and the function $S_{\infty,n_0}\bar{p}_{n_0} \circ \chi^{-1}$ are continuously differentiable in 0.

In order to show (4.72) we consider the maps ξ_{l+r} and ξ_l introduced in the proof of Lemma 4.7. Those maps are reparametrizations of the characteristic map on the rings D_{l+r} and D_l, respectively, such that both maps are defined on D_0. We use the mapping

$$T_{l,r} := \xi_l \circ \xi_{l+r}^{-1} : \chi(D_{l+r}) \to \chi(D_l)$$

to reparametrize limit functions defined on $\chi(D_{l+r})$ and to obtain functions defined on $\chi(D_l) \subset \chi(D'_{n_0})$ where we have the estimate (4.70). In order to analyze the mappings $T_{l,r}$ we need some preparations. First, the estimate (4.68) together with the inverse function theorem shows that there is a constant $C > 0$, independent of the indices $l > n_0$ and $r > 0$, such that

$$\sup_{y \in \chi(D_{l+r})} \|d_y \xi_{l+r}^{-1}\| \leq C(l+r)^{1-m} |\lambda|^{-l-r}. \tag{4.73}$$

Secondly, we proceed similar to (4.65) and (4.67) in Lemma 4.7, but estimate from above, instead of from below, to get a constant C which does not depend on l and $x \in D_0$ such that

$$\|d_x \xi_l(x)\| \leq C|\lambda|^l l^{m-1} \|d_x \psi\|. \tag{4.74}$$

Using the chain rule and both (4.73) and (4.74), we obtain

$$\sup_{y \in \chi(D_{l+r})} \|d_y T_{l,r}\| \leq C\big((l+r)^{1-m}|\lambda|^{-l-r}\big) \cdot \big(|\lambda|^l l^{m-1}\big) \leq C \ |lambda|^{-r} s^r,$$

where C is independent of $l > n_0$ and $r > 0$. Since $S_{\infty,n} p_n \circ \chi^{-1} = S_{\infty,n_0} \bar{p}_{n_0} \circ \chi^{-1} \circ T_{l,r}$ on the ring $\chi(D_l)$, we can apply the chain rule to estimate

$$\begin{aligned}
\sup_{x \in \chi(D_{l+r})} \|d_x(S_{\infty,n} p_n \circ \chi^{-1})\| &\leq \sup_{x \in \chi(D_{l+r})} \|d_x T_{l,r}\| \sup_{x \in \chi(D_l)} \|d_x(S_{\infty,n_0} \bar{p}_{n_0} \circ \chi^{-1})\| \\
&\leq C|\lambda|^{-r} s^r \sup_{x \in \chi(D_l)} \|d_x(S_{\infty,n_0} \bar{p}_{n_0} \circ \chi^{-1})\|,
\end{aligned}$$

where the constant C does not depend on $l > n_0$ and $r > 0$. This proves (4.72), which completes the proof. □

We show the main result of this part.

Proof of Theorem 3.5. We use the ring index n_0 of Lemma 4.7 which guarantees that the estimates of Lemma 4.7 and Lemma 4.8 are valid.

We show that the functions $S_{\infty,i} T_{i-1,0} p_0 \circ \chi$ form a Cauchy sequence in the Banach space $C^1(\chi(D'_{n_0}), \mathbb{R}^d)$. Since this sequence (with each member reparametrized by χ^{-1}) converges to the limit of subdivision in the space $C(D, \mathbb{R}^d)$ according to Theorem 4.6, it also converges to the reparametrized limit of subdivision in the space $C(\chi(D'_{n_0}), \mathbb{R}^d)$. So if the sequence is Cauchy in C^1 its limit agrees with the reparametrized limit of subdivision, which must then be a C^1 function.

In order to show that the sequence $S_{\infty,i} T_{i-1,0} p_0 \circ \chi$ is Cauchy we show that there is a constant C, which does not depend on the level $i \geq n_0$, such that

$$\|(S_{\infty,i+1} T_{i,0} p_0 - S_{\infty,i} T_{i-1,0} p_0) \circ \chi^{-1}\|_{C^1(\chi(D'_{n_0}), \mathbb{R}^d)} \leq C\gamma^i \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0), \tag{4.75}$$

for $\gamma = s^2 \max((\det M)^{-1/2}, \lambda)$, and $s > 1$ chosen such that $\gamma < 1$. If (4.75) is shown, using the geometric series yields the desired statement.

We consider $(i+1)$-st level data q_{i+1} given by

$$q_{i+1} := (T_i - S_i) T_{i-1,0} p_0.$$

According to (4.75), we have to estimate the C^1 norm of the limit function $S_{\infty,i+1} q_{i+1}$ of linear subdivision using S for input data q_{i+1} w.r.t. the characteristic parametrization. For getting fine enough estimates, we split the n_0-th inner area D'_{n_0} into the rings D_n ($n_0 \leq n \leq i$) and the $(i+1)$-st inner area D'_{i+1}. We estimate $S_{\infty,i+1} q_{i+1} \circ \chi^{-1}$ on the domains $\chi(D_n)$ and $\chi(D'_{i+1})$ separately.

We begin with the rings D_n. We fix n with $n_0 \leq n \leq i$. From Lemma 4.5 we get a constant $C > 0$ which does not depend on the ring index n such that

$$\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(T_{n-1,0} p_0) \leq C \lambda^n s^n \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0). \tag{4.76}$$

In Section 3.3 we assumed that the control sets $\mathrm{ctrl}^n(D_n^j)$ of the segments D_n^j have regular combinatorics. Therefore, the limit function w.r.t. linear subdivision using S on the domain D_n is obtained from n-th level data on $\mathrm{ctrl}^n(D_n)$ by means of subdivision on a regular part of the mesh. By the locality of the proximity inequality, the same is true for using T instead of S. Then Lemma 4.3 implies that

$$\begin{aligned}\mathcal{D}_{\mathrm{ctrl}^i(D_n)}(T_{i-1,0} p_0) &\leq C \ \det M^{(n-i)/2} s^{i-n} \mathcal{D}_{\mathrm{ctrl}^n(D_n)}(T_{n-1,0} p_0) \\ &\leq C \ \det M^{(n-i)/2} |\lambda|^n s^i \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0).\end{aligned}$$

For the second inequality we used (4.76). The constants C do not depend on i. The proximity inequality and the above estimate yield

$$\begin{aligned}\mathcal{D}_{\mathrm{ctrl}^{i+1}(D_n)}(q_{i+1}) &\leq C \ \mathcal{D}_{\mathrm{ctrl}^i(D_i)}(T_{i-1,0} p_0)^2 \\ &\leq C \ \det M^{n-i} |\lambda|^{2n} s^{2i} \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0)^2,\end{aligned} \tag{4.77}$$

where the occurring constants do not depend on the index i. We turn to estimating C^1 norms. From the scaling relation and the translation invariance of the scheme S in regular parts of a mesh we get a constant C which is again independent of i and the level n, where $n_0 \leq n \leq i$, such that

$$\|S_{\infty,i+1} q_{i+1}\|_{C^1(D_n,\mathbb{R}^d)} \leq C \ \det M^{i/2} \|q_{i+1}|_{\mathrm{ctrl}^{i+1}(D_{i+1})}\|_\infty. \tag{4.78}$$

These facts together with Lemma 4.7 imply

$$\begin{aligned}\|S_{\infty,i+1}q_{i+1}\circ\chi^{-1}\|_{C^1(\chi(D_n),\mathbb{R}^d)} &\leq C\ |\lambda|^{-n}(\det M)^{-n/2}\|S_{\infty,i+1}q_{i+1}\|_{C^1(D_n,\mathbb{R}^d)}\\ &\leq C\ |\lambda|^{-n}(\det M)^{(i-n)/2}\|q_{i+1}|_{\mathrm{ctrl}^{i+1}(D_{i+1})}\|_\infty.\\ &\leq C\ |\lambda|^{n}s^{2i}(\det M)^{(n-i)/2}\mathcal{D}_{\mathrm{ctrl}^0(D_0)}(p_0)^2.\end{aligned}$$

The constants C do not depend on the indices n and i. For the first inequality we used the estimate (4.60) of Lemma 4.7. The second and the third inequality are a consequence of (4.78) and (4.77), respectively. This proves (4.75) on the rings $\chi(D_n)$ with ring index $n_0\leq n\leq i$.

It remains to consider the $(i+1)$-st inner area D'_{i+1}. We obtain

$$\begin{aligned}\|S_{\infty,i+1}q_{i+1}\circ\chi^{-1}\|_{C^1(\chi(D'_{i+1}),\mathbb{R}^d)} &\leq C\ |\lambda|^{-i}s^i\|q_{i+1}|_{\mathrm{ctrl}^{i+1}(D'_{i+1})}\|_\infty\\ &\leq C\ |\lambda|^{-i}s^i\mathcal{D}_{\mathrm{ctrl}^i(D'_i)}(T_{i-1,0}p_0)^2\\ &\leq C\ |\lambda|^{i}s^{2i}\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0)^2,\end{aligned}$$

where the constants C are independent of i. We use Lemma 4.8 for the first estimate. The second inequality is obtained by applying the local proximity inequality, and Lemma 4.5 gives the last inequality. This estimate proves (4.75) on $\chi(D'_{i+1})$, which completes the proof.

Note that in case we have pure eigenvalues, χ is already invertible on D'_0 and Lemma 4.8 is true for any $n\in\mathbb{N}_0$. So we can choose $n_0=0$ in that case. □

Finally, we show Corollary 3.6.

Proof of Corollary 3.6. It remains to verify the local proximity condition (3.17). This follows directly from [6, Theorem 4] for the projection analogue, from [5, Proposition 7.2] for the log-exp analogue, and from [25, Theorem 1.4] for the intrinsic mean analogue. □

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References

[1] Bump, D. *Lie Groups.* Springer, New York, 2004.

[2] Do Carmo, M. *Riemannian Geometry.* Birkhäuser, Boston, 1992.

[3] DODGSON, N. A. An heuristic analysis of the classification of bivariate subdivision schemes. In *IMA Conference on the Mathematics of Surfaces* (2005), vol. 3604 of *Lecture Notes in Computer Science*, Springer, pp. 161–183.

[4] DONOHO, D. L. Wavelet-type representation of Lie-valued data, 2001. Talk at the IMI Approximation and Computation meeting, Charleston, South Carolina, May 12-17, 2001.

[5] GROHS, P. Smoothness analysis of subdivision schemes on regular grids by proximity. *SIAM J. Numer. Anal. 46* (2008), 2169–2182.

[6] GROHS, P. Smoothness equivalence properties of univariate subdivision schemes and their projection analogues. *Numerische Mathematik 113/2* (2009), 163–180.

[7] GROHS, P. Proximity analysis of nonlinear subdivision schemes and applications to manifold-valued subdivision via the Riemannian center of mass. *SIAM J. Math. Analysis 42* (2010), 729–750.

[8] HABIB, A., AND WARREN, J. Edge and vertex insertion for a class of C^1 subdivision surfaces. *Computer Aided Geometric Design 16* (1999), 223–247.

[9] HAN, B. Computing the smoothness exponent of a symmetric multivariate refinable function. *SIAM J. Matrix Anal. Appl. 24* (2002), 693–714.

[10] HAN, B. Solutions in Sobolev spaces of vector refinement equations with a general dilation matrix. *Adv. Comput. Math. 24* (2006), 375–403.

[11] HAN, B., AND JIA, R.-Q. Multivariate refinement equations and convergence of subdivision schemes. *SIAM J. Math. Anal. 29* (1998), 1177–1199.

[12] JIA, R.-Q. Approximation properties of multivariate wavelets. *Math. Comput. 67* (1998), 647–665.

[13] KARCHER, H. Riemannian center of mass and mollifier smoothing. *Comm. Pure Appl. Math. 30* (1977), 509–541.

[14] KOBBELT, L. $\sqrt{3}$-subdivision. *Computer Graphics Proceedings (SIGGRAPH)* (2000), 103–112.

[15] OSWALD, P. Designing composite triangular subdivision schemes. *Comput. Aided Geom. Design 22* (2005), 659–679.

[16] OSWALD, P., AND SCHRÖDER, P. Composite primal/dual $\sqrt{3}$-subdivision schemes. *Comput. Aided Geom. Design 20* (2003), 135–164.

[17] PETERS, J., AND REIF, U. The simplest subdivision scheme for smoothing polyhedra. *ACM Trans. Graph. 16* (1997), 420–431.

[18] Peters, J., and Reif, U. *Subdivision Surfaces.* Springer, Berlin, 2008.

[19] Reif, U. A unified approach to subdivision algorithms near extraordinary vertices. *Comput. Aided Geom. Design 12* (1995), 153–174.

[20] Sauer, T. Differentiability of multivariate refinable functions and factorization. *Adv. Comput. Math. 25* (2006), 211–235.

[21] Triebel, H. *Interpolation Theory, Function Spaces, Differential Operators.* North-Holland, Amsterdam, 1978.

[22] Ur Rahman, I., Drori, I., Stodden, V. C., Donoho, D. L., and Schröder, P. Multiscale representations for manifold-valued data. *Multiscale Mod. Sim. 4* (2005), 1201–1232.

[23] Wallner, J. Smoothness analysis of subdivision schemes by proximity. *Constr. Approx. 24* (2006), 289–318.

[24] Wallner, J., and Dyn, N. Convergence and C^1 analysis of subdivision schemes on manifolds by proximity. *Comput. Aided Geom. Design 22* (2005), 593–622.

[25] Weinmann, A. Nonlinear subdivision schemes on irregular meshes. *Constr. Approx. 31* (2010), 395–415.

[26] Xie, G., and Yu, T. P.-Y. Smoothness equivalence properties of general manifold-valued data subdivision schemes. *Multiscale Mod. Sim. 7* (2008), 1073–1100.

[27] Xie, G., and Yu, T. P.-Y. Smoothness equivalence properties of interpolatory Lie group subdivision schemes. *IMA Journal of Numerical Analysis (to appear)* (2010).