INTERPOLATORY MULTISCALE REPRESENTATION FOR FUNCTIONS BETWEEN MANIFOLDS

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ABSTRACT. We investigate interpolatory multiscale transformations on irregular combinatorics which handle manifold-valued data. We characterize the Hölder-Zygmund smoothness of a function between manifolds in terms of the coefficient decay w.r.t. this multiscale transform.

Keywords: Nonlinear subdivision, irregular vertex, interpolatory multiscale transform.

1. INTRODUCTION

Linear subdivision schemes and multiscale transforms derived from subdivision are, for example, used in computer graphics and geometric modeling [29] and also in the numerical solution of PDEs [11, 10]. In recent years, nonlinear subdivision schemes and corresponding multiscale transforms have gained a lot of interest. To get an impression of the diversity of this field the reader is referred to [9, 22, 23, 26] and the references therein. In this article we stick to the nonlinear geometric setting: Geometric subdivision and geometric multiscale transforms handle data in nonlinear geometries such as Lie groups, symmetric spaces, or Riemannian manifolds. Examples are the Euclidean motion group, hyperbolic space, Grassmannians or the space of positive definite matrices.

In [4], D. Donoho analyzes linear *interpolatory wavelet transforms*. In particular he characterizes smoothness properties of a function by decay properties of the so-called detail coefficients which are derived from the function via the transformation. Interpolatory transforms can also be defined in a reasonable manner in the setting of geometric subdivision [22]. In [7], Grohs and Wallner show an analogue of Donoho's result for the class of Hölder-Zygmund functions in the geometric setting. More precisely, they consider a continuous function f defined on $\mathbb{R}^n$ with values in a manifold M. This function is sampled on the grid $2^{-i}\mathbb{Z}^n$ to obtain a grid function f_i. A geometric subdivision scheme T is applied to f_i and a (generalized) difference $f_{i+1} \ominus Tf_i$ between this prediction Tf_i and the (finer) sample $f_{i+1} = f|_{2^{-i-1}\mathbb{Z}^n}$ gives the i-th level detail coefficients d_i. The function f is a Hölder-Zygmund function of order α, if and only if the detail coefficients d_i decay with $O(2^{-\alpha i})$ as $i \to \infty$.

In this article we treat manifold-valued functions defined on a two-dimensional manifold. We consider a multiscale transform where both the choice of sample points and the prediction operator are based on nonlinear geometric subdivision. As closed 2-manifolds with non-zero Euler characteristic cannot be covered with regular quad meshes or triangular meshes, we must be able to process irregular combinatorics.

The paper is organized as follows. We start out by gathering the necessary information on linear and geometric subdivision (with emphasis on the situation near irregular vertices).

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Then we define an interpolatory multiscale transform. Afterwards we recall results for geometric nonlinear subdivision near irregular (or, synonymously, extraordinary) vertices. Then we have a look at interpolatory wavelet transforms on regular grids.

The rest of the paper it devoted to the characterization of Hölder-Zygmund functions fetween manifolds in terms of the detail coefficient decay, in particular near irregular points. The main result of the paper is Theorem 2.3. Its formulation is somewhat involved which is due to the fact that the subdominant eigenvalue of the subdivision matrix enters the scene when formulating the decay condition of the detail coefficients $d_i = f_{i+1} \ominus Tf_i$. However, for certain schemes like the modified butterfly scheme [5, 31] the statement simplifies. We have the following qualitative statement:

Corollary (Theorem 2.3 for the Butterfly scheme)**.** *Let M be a smooth manifold and let N be a closed surface. For a continuous function $f : N \to M$ and any positive γ, which is smaller than the smoothness of the butterfly scheme on regular meshes we have the equivalence*

$$f \in \mathrm{Lip}(\gamma) \quad \textit{if and only if} \quad \|d_i\|_\infty \leq C 2^{-i\gamma}.$$

1.1. **Linear and Geometric Subdivision Schemes.** We consider meshes of the form (K, p), where $K = (V, E, F)$ represents the abstract combinatorics and $p : V \to M$ is a vertex based positioning function. A *linear subdivision scheme* S maps a given input mesh (K_0, p_0) to a (finer) output mesh (K_1, p_1) by acting as a linear operator on the linear space of positioning functions. This works only if M is a vector space. S consists of a *topological* refinement rule generating the new combinatorics K_1 from K_0, and a *geometric* refinement rule generating the new positioning function p_1. In order not to introduce additional notation we consider *primal triangular* and *primal quadrilateral* topological rules. We assume that the linear scheme is affinely invariant: This means that the position of a new vertex $p_1(w)$ is computed as an affine average of old positions as follows:

$$p_1(w) = \sum\nolimits_v \alpha_{v,w} p_0(v) = x(w) + \sum\nolimits_v \alpha_{v,w}(p_0(v) - x(w)), \tag{1.1}$$

where $\sum_v \alpha_{v,w} = 1$ and $x(w)$ is some arbitrary point. The second equality is a consequence of the affine invariance of S. The point $x(w)$ is called *base point* and becomes only important in the nonlinear setting. We assume that $\alpha_{v,w} \neq 0$ only if v is in a neighborhood of w of a certain globally fixed size. Furthermore, the averaging rules shall only depend on the combinatorics of a mesh neighborhood of w of globally fixed size. This is the same setting as is used in [28]. A subdivison scheme S is *interpolatory* if $V_i \subset V_{i+1}$ and old vertex positions are not changed during the subdivision process. In that case subdivision adds new vertices to the existing ones.

Starting with the linear rule (1.1) as a template, we explain how to construct a scheme which works in a manifold. We retain the topological refinement rule and modify the geometric rule so as to work in a manifold. We begin with a subdivision scheme for Riemannian manifolds.

Intrinsic mean subdivision: Observe that in Euclidean space the weighted center of mass $p_1(w)$ in (1.1) is the minimizer of a quadratic function:

$$p_1(w) = \mathrm{argmin}_q \sum\nolimits_v \alpha_{v,w} \|p_0(v) - q\|_2^2.$$

By replacing the Euclidean distance by the Riemannian distance, we obtain the modified rule

$$p_1(w) = \operatorname{argmin}_q \sum\nolimits_v \alpha_{v,w} \operatorname{dist}(p_0(v), q)^2 \tag{1.2}$$

which applies to data in a Riemannian manifold. Existence and uniqueness of $p_1(w)$ are guaranteed if the distance between contributing old vertex positions $p_0(v)$ is small enough. The precise bounds depend on the sectional curvature of the Riemannian manifold under consideration [12]. This minimizer is called (weighted) *Riemannian center of mass* or *intrinsic mean*. Using the rule (1.2) naturally preserves the symmetries present in the coefficients $\alpha_{v,w}$. We have the following nice property:

$$\sum\nolimits_v \alpha_{v,w} \ \exp^{-1}_{p_1(w)}(p_0(v)) = 0. \tag{1.3}$$

Here exp is the Riemannian exponential mapping. (1.3) implies

$$p_1(w) = \exp_{p_1(w)} \big(\sum\nolimits_v \alpha_{v,w} \ \exp^{-1}_{p_1(w)}(p_0(v)) \big). \tag{1.4}$$

If the old vertex positions $p_0(v)$ sit in a small enough Riemannian ball, the balance condition (1.3) even characterizes the center of mass (1.2). This property could also serve as a definition if no distance is available, like in a Lie group.

Log-exp subdivision: By replacing $p_1(w)$ in the right hand side of (1.4) by some base point $x(w)$ we get the rule

$$p_1(w) = \exp_{x(w)} \big(\sum\nolimits_v \alpha_{v,w} \ \exp^{-1}_{x(w)}(p_0(v)) \big), \tag{1.5}$$

which is a direct analogue of (1.1) as shall be explained in more detail below. For our purposes the choice of base points is rather arbitrary: $x(w)$ should just be chosen to lie in a neighborhood (of globally fixed size) of w. Subdivison using the rule (1.5) is called *log-exp subdivision* [22].

Note that by (1.4) intrinsic mean subdivision is an instance of log-exp subdivision with a very special choice of base points, namely the mean itself. Comparing (1.5) with (1.1), we see that the operation 'point + vector' is replaced by the exponential mapping and that the operation 'point − point' is replaced by the inverse of exp . For p, q in a Riemannian manifold and a tangent vector v, we let

$$p \oplus v = \exp_p(v) \quad \text{and} \quad q \ominus p = \exp_p^{-1}(q).$$

Then (1.5) arises from (1.1) by replacing + and − by $\oplus$ and $\ominus$, respectively.

Starting from this interpretation we can take the following viewpoint for constructing *geometric analogues* of subdivision schemes of which (1.2) and (1.5) are examples. A geometric analogue T of the linear scheme S retains the topological rule. The geometric rule is adapted to work in nonlinear geometries by replacing vector space operations by suitable substitutes. Various such constructions for different geometries, including Lie groups and symmetric spaces, have been discussed in detail, see e.g. [22, 23, 24]. It is common to virtually all geometric schemes that in general the functions used in their construction are not globally defined, but their existence is only guaranteed locally. This translates to the fact that the input data have to be dense enough to ensure that the geometric scheme is well-defined. This also has been extensively discussed e.g. in [23, 24].

General bundle framework: We briefly recall a general framework set up in [7] which applies to the examples above. It is assumed that the manifold M is the base space of a smooth vector bundle $\pi : E \to M$ with a smooth bundle norm (e.g. in a Lie group the

trivial bundle with the Lie algebra as fiber and some canonically extended norm on the Lie algebra, or the tangent bundle of a Riemannian manifold with the norm induced by the Riemannian scalar product). The substitutes of addition and subtraction are given by an operation $\oplus : E \to M$, which is defined in a neighborhood of the zero section of the bundle, and an operation $\ominus : M \times M \to E$, which is defined near the diagonal. (E.g. the Lie group exponential or the Riemannian exponential and their inverses.) Furthermore, the consistency conditions $y \ominus x \in \pi^{-1}(\{x\})$ and $x \oplus (y \ominus x) = y$ have to be fulfilled. Then the geometric analogue of (1.1) w.r.t. this bundle is given by

$$p_1(w) = x(w) \oplus \sum\nolimits_v \alpha_{v,w}(p_0(v) \ominus x(w)). \tag{1.6}$$

Because of (1.4), intrinsic mean subdivision can be interpreted as a log-exp analogue with a special choice of base points and thus fits into this framework.

1.2. **Definition of a Multiscale Transformation for Geometric Data.** In the following let N be a two-dimensional smooth *domain* manifold, and let M be a smooth *target* manifold of arbitrary dimension. We explain a way of sampling continuous functions from N to M: Consider a mesh (K_0, p_0) which covers N. We use an interpolatory subdivision scheme T', which processes data in N and which is analogous to a linear scheme S. By applying T', we get meshes $(K_1, p_1), (K_2, p_2), \ldots$. By construction, these meshes have subdivision connectivity. The (realized) vertex sets $X_i = p_i(V_i)$ in N are nested.

We assume that never two (realized) vertices $p_i(v)$ and $p_i(w)$ coincide, i.e., we assume that p_i is injective.Sufficient conditions for injectivity are given in Section 2.1.

We propose the following discrete interpolatory multiscale transform: We point-sample a continuous function $f : N \to M$ on X_i and let

$$f_i = f|_{X_i}.$$

So f_i is an M-valued fuction defined in the discrete subset $X_i \subset N$.

To define a prediction operator T we use another interpolatory analogue T'' of S which this time works in M. T'' is applied to the mesh $(K_i, f \circ p_i)$ whose realized vertex set is $f_i(X_i)$. The result is a mesh (K_{i+1}, g_{i+1}) where g_{i+1} has values in M. By our assumption on

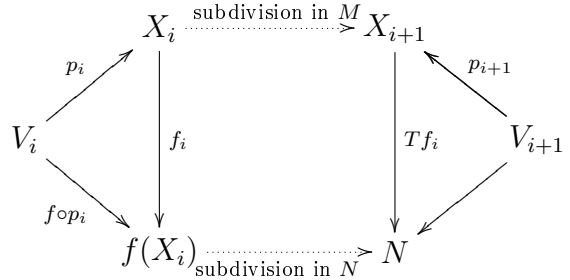

FIGURE 1. Definition of the prediction operator for a multiscale transform based on interpolatory geometric subdivision.

the injectivity of p_{i+1}, the function $g_{i+1} \circ p_{i+1}^{-1} : X_{i+1} \to M$ is well defined. We define the prediction operator T by

$$Tf_i = g_{i+1} \circ p_{i+1}^{-1}.$$

Using the geometric operation $\ominus$ pointwise, detail coefficients are defined by

$$d_i = f_{i+1} \ominus Tf_i.$$

Our multiscale transform is now defined by

$$R : f \to (f_0, d_0, d_1 \ldots). \tag{1.7}$$

Note that the well-definedness of the transform depends on the well-definedness of the subdivision operators T' and T'', which in general can only be guaranteed for dense enough input data. This translates to the fact that we cannot arbitrarily choose the coarsest level for sampling (as in the linear case), but there is a bound on the maximal 'zoom out'. It turns out, however, that the guaranteed theoretical bounds are very pessimistic in contrast to what can be observed in practice.

In applications, we have the following finite version of the transform. It reads

$$R_n : f_n \to (f_0, d_0, \ldots, d_{n-1}). \tag{1.8}$$

A special case occurs if M is a vector space and T'' is a linear scheme. Then the multiscale transform is linear.

On the other hand, if $N = \mathbb{R}^2$ and the initial covering of N is given by the $\mathbb{Z}^2$ lattice, choosing T' as an interpolatory linear scheme which reproduces linear functions yields the multiscale transform defined in [22].

1.3. **Combinatorial Setup.** The analysis of subdivision schemes w.r.t. local properties such as smoothness or convergence (on compact sets) splits into two parts: The first one is to consider *regular* meshes and analyze the properties of the scheme for those meshes. Since subdivision does not introduce additional irregular vertices as subdivision progresses, an irregular vertex gets surrounded by an arbitrary large regular mesh with the irregular vertex as its only singularity. The assumed locality of the scheme guarantees that away from an irregular vertex one only has to deal with a regular mesh, and near an irregular vertex one can deal with an unbounded mesh with only one central singularity. The second part is to analyze the latter situation.

A regular mesh is typically identified with a function on the domain $V_0 = \mathbb{Z}^2$, where $\mathbb{Z}^2$ is naturally embedded into the domain $D = \mathbb{R}^2$. Here the combinatorics is understood implicitly. A k-regular mesh (one central irregular vertex of valence k) is typically identified with a function on a discrete subset of the following domain D : D is obtained by cyclically gluing k copies of a sector Ω in the plane with opening angle 90° in the quad case (or 60° in triangular case), i.e.,

$$D = \Omega \ \times \ \mathbb{Z}_k,$$

where $\mathbb{Z}_k$ are the integers modulo k. The gluing is done as follows: In each sector we have polar coordinates (x, ϕ) where $0 \leq \phi \leq 90^\circ$ (60°, resp.). The points $(x, 90^\circ)$ of the first sector and the points $(x, 0^\circ)$ of the second sector are identified, and so on, where the points $(x, 90^\circ)$ in the k-th sector and $(x, 0^\circ)$ in the first sector are also identified. In the triangular case, $(x, 90^\circ)$ is replaced by $(x, 60^\circ)$. We refer to Figure 2 for a visualization. The domain D is an abstract space which turns into a metric space by defining the distance of points by the length of the shortest path which connects them, with the metric in the single sectors being that of $\mathbb{R}^2$.

We identify a k-regular mesh with a function on the discrete subset V_0 of the domain D which we obtain as follows (see Figure 2): If the scheme is quad-based, we let Σ be the unit square in Ω. If the scheme is triangle-based, Σ stands for the equilateral triangle of length one in Ω. We consider the tiling of Ω with proto-tile Σ. The corners of these tiles constitute the restriction of V_0 to Ω. Forming the union over all copies of Ω, we obtain V_0. So a k-regular mesh can be seen as function on V_0, and iterated subdivision produces functions on $V_1, V_2, \ldots$, where $V_i = 2^{-i} V_0$. Hence, for any i, a subdivision scheme T induces an operator T_i which maps functions on V_i to functions on V_{i+1}. We use the notation $T_{i,j}$ which is short for $T_i \cdots T_j$ (If $i < j$, let $T_{i,j}$ be the identity.).

For analysis purposes, the domain D is partitioned into so-called rings $D_i, i \geq -1$ (see again Figure 2): For nonnegative i, we let

$$D_i = n(2^{-i}\Sigma \backslash 2^{-i-1}\Sigma) \times \mathbb{Z}_k, \quad \text{and} \quad D_{-1} = \Omega \backslash n\Sigma \times \mathbb{Z}_k. \tag{1.9}$$

Here $n \geq 1$ is an integer which depends on the subdivision scheme under consideration. It must be chosen so large that the limit of subdivision on D_0 is obtained from its control set in V_0 by means of the 'regular mesh' subdivision rules. A typical value is $n = 4$. For details we refer to [25]. D' denotes the union of all copies of $n\Sigma$, or in other words D' is the union of all rings D_i ($i \geq 0$) and the central point 0.

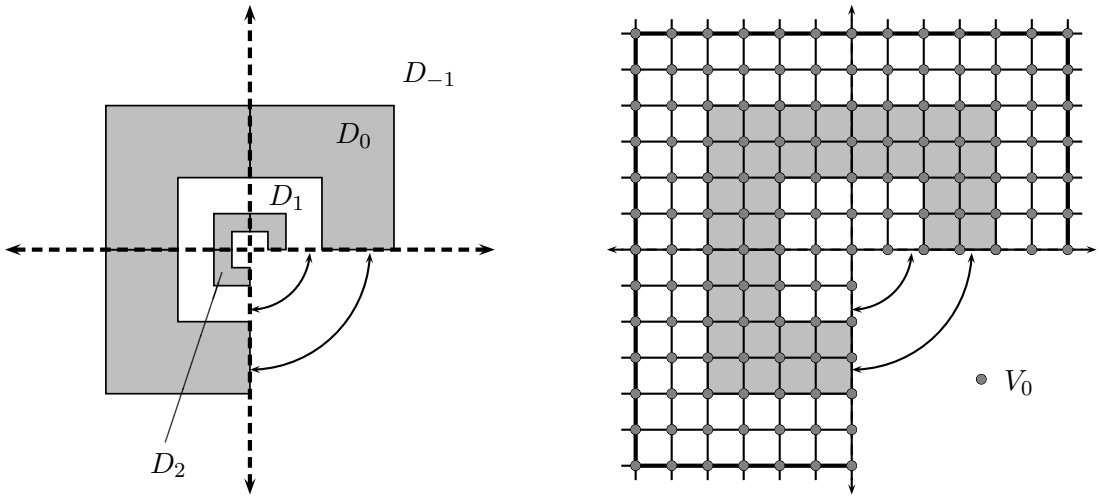

FIGURE 2. Parametrization near an extraordinary vertex of valence 3 in case of a quad mesh. *Left:* The domain D is obtained by gluing three quadrants together. The first three rings D_0, D_1 and D_2 are visualized. *Right:* The set V_0 of vertices contained in D is visualized for parameter $n = 4$ as needed for Kobbelt's interpolatory 4-point scheme. The thick line bounds the level 0 control set of the ring D_0.

1.4. **Rigorous Analysis Setup and Results for Geometric Subdivision.** One way of analyzing local properties of a geometric scheme T is to go to charts and then, in these charts, to compare it to the linear scheme S it is derived from. So let us first consider the case where data is in $M \subset \mathbb{R}^d$. One can think of M as the image under a chart. We quantify the phrase 'dense' which often occurs in our theorems: We define the *class* $P_{M,\delta}$ *(*$M \subset \mathbb{R}^d, \delta > 0$*)* to be meshes whose vertices sit in M and whose diameter of faces is bounded by δ. To compare schemes T and S we say that S and T are *in local proximity w.r.t.* $P_{M,\delta}$, if, for all input data $(K_0, p_0) \in P_{M,\delta}$, the positioning functions p_1^T and p_1^S fulfill

$$\|p_1^S(w) - p_1^T(w))\|_{\mathbb{R}^d} \leq C \sup_{v_1, v_2 \in \mathrm{supp}(w)} \|p_0(v_1) - p_0(v_2)\|^2, \tag{1.10}$$

where the support 'supp(w)' of the stencil of S at w are those old vertices which contribute to $p_1^S(w)$. Here C is a constant independent of input p_0 and w. In the general case that M is a manifold with local coordinate charts ϕ_i, definition (1.10) is applied to the coordinate representation of T, i.e., T is replaced by $\phi_i \circ T \circ \phi^{-1}$.

The proof of a local property, like smoothness of T, runs as follows: First, one has to show that the coordinate representation of T and the linear scheme S it is derived from fulfill

the proximity condition. The second part is to prove that proximity allows us to transfer the desired property from the linear scheme S to T. For the second part, the proximity condition is the only assumtion on T, it need not be a geometric analogue.

We gather information on convergence and smoothness: We say that a subdivision scheme T *converges for regular/k-regular meshes in* $P_{M,\delta}$, if for bounded input $p_0 \in P_{M,\delta}$, iterated subdivision is defined and there is a continuous function f on D such $\|f|V_i - T_{i,0}p_0\|_\infty \to 0$ as $n \to \infty$. The function f is called the limit function. We write $f = T_{\infty,j}p_j$ for data $p_j = T_{j-1,0}p_0$.

It has been pointed out in [25] that a large class of geometric schemes meet proximity conditions in the case of irregular combinatorics. In the same paper we prove that the limits of those geometric analogues converge and are C^1. Indeed, we show that if a scheme T is in proximity with a linear scheme S, then T produces C^1 limits, provided S meets the following conditions:

(1) On regular meshes, the scheme S is stable in the sense that the operator assigning the limit function to data is a lower bounded operator w.r.t. from the space of bounded data to the space of bounded continuous functions, both equipped with the sup-norm. (This condition is automatically fulfilled if S is interpolatory.) S produces C^1 limits on regular combinatorics.

(2) There is a matrix A (here called the *subdivision matrix*) that maps data on an n-ring around the extraordinary vertex of a k-regular mesh to the corresponding n-ring of the subdivided mesh. This n-ring is large enough to control the limit function on D_0 (see Figure 2).

(3) For any valence k, the subdivision matrix A has the single dominant eigenvalue 1 and subdominant eigenvalue $\lambda \in]0, 1[$ whose algebraic and geometric multiplicity is 2. The characteristic map χ, defined below, is regular and injective.

The *characteristic map* [20, 19, 28] is the limit function of subdivision with S on a k-regular mesh for the following two-dimensional input: The first components consist of one subdominant eigenvector and the second component consists of another linearly independent subdominant eigenvector. In this paper we always assume that S is interpolatory, and that S fulfills the above requirements.

Near extraordinary vertices the smoothness is measured w.r.t. characteristic parametrization, i.e., $T_{\infty,0}p_0 \circ \chi^{-1}$ is considered. There is also an interpretation as smoothness of a mapping from a certain differentiable manifold Q. Q is obtained by imposing a smoothness structure on the mesh by considering it as a topological space in the cannonical way and using the characteristic maps as charts (those are defined in each 1-ring N_v neighborhood of a vertex v):

$$\chi_v : N_v \subset Q \to \mathbb{R}^2$$

. Q is as smooth as the limits of S on the regular mesh. The *smoothness of a scheme* is the smoothness of limit functions w.r.t. this differentiable structure. For details, we refer to [1].

1.5. **Linear and Geometric Interpolatory Wavelets.** We briefly summarize the results obtained for both linear and geometric interpolatory wavelet transforms for functions on $\mathbb{R}^n$. Here a continuous function $f : \mathbb{R}^n \to M$ (or $\mathbb{R}$) is point-sampled: We let $f_i = f|_{2^{-i}\mathbb{Z}^n}$. Detail coefficients are computed as $d_i = f_i \ominus Tf_{i-1}$ where T is a geometric scheme. Then the transform reads $f \to (f_0, d_0, d_1, \ldots)$. The corresponding linear transform is obtained by replacing T by a linear scheme S, and $\ominus$ by $-$.

In this context smoothness of a function is measured by its membership in the Hölder-Zygmund class Lip_α. For $\alpha < 2$, a continuous bounded function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to Lip_α, if the second differences $\Delta_h^2 f = f(x+h) - 2f(x) + f(x-h)$ obey the inequality $\|\Delta_h^2 f\|_\infty \le Ch^\alpha$ for some $C > 0$ and all $h \in \mathbb{R}^n$. For $\alpha \ge 2$, write $\alpha = k + \beta$ with a positive integer k and $0 < \beta < 2$. Then f belongs to Lip_α if all k-th order parial derivatives $D^\mu f$ are contained in Lip_β, where μ is a multi-index of degree k.

The following theorem is part of the results of [4] and the result of [7]:

Theorem 1.1. *Let S be a linear interpolatory subdivision scheme on the regular mesh which produces Lip_α limits, and assume that f is a continuous function on $\mathbb{R}^d$ with image contained in a compact subset.*

Then, for $\gamma < \alpha$, $f \in \mathrm{Lip}_\gamma$ if and only if the coefficients d_i w.r.t. the linear scheme decay as $O(2^{-\gamma i})$, i.e., there is $C > 0$ such that $2^{\gamma i}\|d_i\|_\infty \le C$ for all i.

Assume furthermore that T is a geometric analogue of S, and that f_0 is dense enough such that the geometric version of the transform is defined. Then the detail coefficients w.r.t. T also decay as $O(2^{-\gamma i})$ if and only if $f \in \mathrm{Lip}_\gamma$.

2. Analysis of the Transformation

2.1. **Results and Examples.** In order not to introduce additional technical problems, we formulate our results for the case when N is compact. However, considering compact sets N and using a local definition of Hölder-Zygmund functions seems a straightforward way to generalize the results to non-compact N.

Our main theorem is Theorem 2.3. Its formulation needs the following notions: the smoothness index of a linear subdivision scheme, Hölder-Zygmund functions between manifolds, a certain non-degeneracy property referring to a mesh covering a manifold, and the quantities $\|d_i\|_{i,\gamma}$ ($i \in \mathbb{N}_0$) which encode the decay of the coefficients under the transformation (1.7). We define these objects first and then state the theorem.

Non-degeneracy Property of a Covering Mesh. Consider the initial mesh covering the manifold N in Section 1.2. We have assumed in Section 1.2 that for both the initial mesh and its subdivided meshes no two abstract vertices coincide in their realization in N. For analysis purposes, we consider the mapping κ from the manifold Q (defined at the end of Section 1.4) to N, which is given as the limit of subdivision. We request the following *non-degeneracy property*:

$$\kappa : Q \to N \text{ is regular and injective.} \tag{2.1}$$

Obviously, this property guarantees that no vertices of the initial mesh or its subdivided meshes coincide in N. Furthermore, it guarantees that κ is onto, and thus invertible. This follows e.g. from degree theory [14][1]

[1]For the reader's convenience we give the following short direct argument: Consider a curve $\gamma : [0,1] \to N$ connecting a point $x = \gamma(0)$ in the image $\kappa(Q)$ and an arbitrary point $y = \gamma(1)$ in N. Consider the maximal parameter t_0 such that for all smaller parameters $t < t_0$ the curve $\gamma([0,t])$ stays in $\kappa(Q)$. The compactness of N implies that $\gamma([0,t_0]) \subset \kappa(Q)$. So there is $p \in Q$ with $\kappa(p) = \gamma(t_0)$ and κ is a local diffeomorphism. Now, if t_0 were not 1, the inverse function theorem and the continuity of γ would guarantee that there is a neighborhood U of $\kappa(p) \subset \kappa(N)$ and $\varepsilon > 0$ such that $\gamma([t_0 - \varepsilon, t_0 + \varepsilon]) \subset U$. This is a contradiction and therefore κ is onto.

If N has non-zero Euler characteristic, we can weaken (2.1) by dropping the injectivity assumption which then is fulfilled automatically. Again, this a consequence of degree theory [14][2].

Corollary 2.14 in [25] yields a way to infer the regularity of κ from properties of intial data p_0 using the regularity of the according limit of S (if p_0 does not satisfy this condition, there is still the chance that $p_1, p_2, \ldots$ do).

Definition of the decay measure $\|d_i\|_{i,\gamma}$. In contrast to the very simple decay conditions in Theorem 1.1 we have somewhat more involved, but still simple conditions near extraordinary vertices. To formulate these conditions we need the following notation:

The *control set* $\mathrm{ctrl}^i(U)$ of $U \subset D$ on subdivision level i consist of those vertices in the i-th mesh which influence the limit function on U in the sense that a change of data on $\mathrm{ctrl}^i(U)$ changes the limit function on U. A more detailed definition is given in [28].

For fixed i, we split the domain D into the rings D_j $(0 \le j < i)$ and the inner area $D \backslash (D_0 \cup \ldots \cup D_{i-1})$. For their i-th level control sets we use the notation

$$\begin{aligned} V_i^j &= \mathrm{ctrl}^i(D_j), && j < i, \\ V_i^i &= \mathrm{ctrl}^i(D \backslash (D_0 \cup \ldots \cup D_{i-1})). \end{aligned} \tag{2.2}$$

The corresponding subsets of X_i (defined at the beginning of Section 1.2) are denoted by $X_i^j = p_i(V_i^j)$. We take the difference $d_{i-1} = f_i \ominus T f_{i-1}$ and componentwise measure its size with the bundle norm. Then we define

$$\|d_i\|_{i,\gamma} = \max_j (\lambda^{-j} 2^{i-j})^\gamma \|s_i|_{X_i^j}\|_\infty, \qquad \text{where } s_i(x) = \|d_i(x)\|. \tag{2.3}$$

Here λ is the subdominant eigenvalue of the subdivision matrix A. It turns out that this is the appropriate quantity to measure the detail coefficient decay near extraordinary vertices with.

Note that our definition is essentially a weighted sup-norm, where the weights depend on the 'distance' to an extraordinary vertex.[3]

The definition of $\|\cdot\|_{i,\gamma}$ naturally extends to an arbitrary mesh and the corresponding subdivided meshes: Near extraordinary vertices, we locally use the above definition and obtain a global definition by 'gluing'. Therefore, we do not introduce complicated notation for that situation.

Smoothness Index of a Linear Subdivision Scheme. We assume that S fulfills the requirements of Section 1.4. Let ν be the smoothness index of S on regular meshes, i.e., the maximal number such that S produces Lip_γ limits for all $\gamma < \nu$. Now we consider the subdivision matrix A for a valence k vertex. We order the eigenvalues according to their

[2] As above, we give a short argument for the reader's convenience: By the regularity of κ and the compactness of Q, it follows that κ is a smooth finite covering. Then the Euler characteristics of the covering space Q must be a multiple of that of N. But this is a contradiction to the fact that the manifolds N and Q are homeomorphic.

[3] To make this precise, for the position of a vertex x near an extraordinary vertex in N, we find an according ring D_i. (If it lies on the boundary between two rings, take the minimal index.). If we use the weight $\lambda^{-j} 2^{i-j}$ for j-th level data $(j \ge i)$ on x we end up with a weighted sup-norm for the j-th level 'sequences' (For inner vertices x which lie in no ring or a ring with $j < i$ use the weight λ^{-j}). Then if we componentwise apply the exponents γ, we end up with an equivalent description of the above situation. However, the above definition is more suitable for the proofs later on.

modulus by $1, \lambda, \lambda, \mu_3, \mu_4, \ldots$. Then we let $\nu' = \min(\log_\lambda |\mu_3|, 2)$ (subdivision schemes with $\log_\lambda |\mu_3| > 2$ are not desirable anyway [18]). We call

$$\omega = \min(\nu, \nu') \tag{2.4}$$

the *smoothness index* of S near an extraordinary vertex of valence k. For a general mesh, take the minimum of the smoothness indices of all extraordinary vertices.

On the Definition of Hölder-Zygmund Classes for Functions between Manifolds. Here we first follow Triebel [21] to define Hölder-Zygmund functions from N to $\mathbb{R}$. We equip N with an auxiliary Riemannian structure. We consider finitely many exponential charts $\exp_{p_i}^{-1}$ (whose images are balls of the same radius r) covering N and a subordinate C^∞ partition of unity $\{\varphi_i\}$. We say a continuous function $f : N \to \mathbb{R}$ belongs to the *Hölder-Zygmund class* $\mathrm{Lip}_\alpha(N, \mathbb{R})$ if $(f\phi_i) \circ \exp_{p_i}$ is a $\mathrm{Lip}(\alpha)$-function on $\mathbb{R}^2$, if we consider it extended by 0 outside the ball of radius r.

Note that this definition does not depend on the chosen Riemannian structure. It also does not depend on the chosen centers of the balls, or on the radius r, or the partition of unity [21]. So the imposed Riemannian structure is only a tool for defining the Hölder-Zygmund Classes, and does not prejudice the subdivision scheme we are going to employ: If N is, for example, a Lie group we can still use a Lie group scheme.

We are going to define the class $\mathrm{Lip}_\alpha(N, M)$ where both N and M are smooth manifolds and N is compact. We equip both N and M with an auxiliary Riemannian structure.

Definition 2.1. *Suppose that finitely many open geodesic balls $B(x_i, r)$ cover N such that each $f(B(x_i, r))$ is contained in one of the finitely many balls $B(y_j, R)$, where the balls $B(y_j, R)$ cover* $\operatorname{im} f$. *Assume that the partition of unity $\{\varphi_i\}$ is subordinate to the balls $B(x_i, r)$. We define the class of Hölder-Zygmund functions $f : N \to M$ by*

$$f \in \mathrm{Lip}_\gamma(N, M) \quad \Longleftrightarrow \quad f_i \in \mathrm{Lip}_\gamma(\mathbb{R}^m, \mathbb{R}^n),$$

where f_i is obtained from $(g_i\varphi_i) \circ \exp_{x_i} : B(0, r) \to \mathbb{R}^n$ by extending with 0 outside the ball, and $g_i = \exp_{y_j}^{-1} \circ f|_{B(x_i, r)}$.

Note that in the above definition, the main purpose of introducing the Riemannian structure is to obtain nice charts. Concerning well-definedness we have the following statement, whose proof is given later on.

Proposition 2.2. *The definition of* $\mathrm{Lip}_\gamma(N, M)$ *does not depend on the imposed Riemannian structures, the particular choice of balls, or the partition of unity.*

We formulate our main result:

Theorem 2.3. *Let S be an interpolatory linear scheme as in Section 1.4 with smoothness index $\omega > 1$ on the mesh combinatorics K. Assume furthermore that the two schemes T' and T'' (acting in N and M, resp.,) both fulfill the local proximity conditions* (1.10) *w.r.t. S. Assume that the initial mesh covering N needed to define the multiscale transform* (1.7) *has the non-degeneracy property* (2.1). *Then smoothness of a continuous function $f : N \to M$ is related to the decay of detail coefficients d_i w.r.t. this multiscale transform as follows:*

$$f \in \mathrm{Lip}_\gamma(N, M) \quad \text{if and only if} \quad \sup_{i \in \mathbb{N}_0} \|d_i\|_{i,\gamma} \le C \tag{2.5}$$

for $0 < \gamma < \omega$. Here $\|\cdot\|_{i,\gamma}$ is defined by (2.3).

In Section 1.2 we already encountered the fact that nonlinear subdivision schemes are in general only defined for dense enough input. By choosing a high enough index i_0, the samples of f on all levels X_i with $i \geq i_0$ are dense enough such that the multiscale transform is well defined if we start on level i_0 instead of level 0. Then the statement of the theorem holds if we choose the i_0-th level mesh as initial mesh. As the statement is an asymptotic one in i, the initial level i_0 does not matter anyway.

We want to point out that by considering N as a smooth (meaning C^∞) manifold, Theorem 2.3 does not apply to the case when N itself is a subdivision surface in $\mathbb{R}^3$. The central technical reason for that is our use of geodesic balls in the definition of the Hölder-Zygmund classes. This is done to obtain 'nice' chart neighborhoods. However, a subdivision surface already brings nice chart neighborhoods. Although we omit this case in this paper to avoid further technical complications, we strongly conjecture that the above theorem is also true when N is a subdivision surface.

Modifications of our proofs would also work for C^1 schemes with $\omega = 1$. However, this would produce an additional case in most situations which we want to omit. Furthermore, we want to point out that we do not know how to prove the above theorem if the scheme is not C^1, or $\omega < 1$.

For the geometric situation we have the following result:

Corollary 2.4. *If T' and T'' are geometric (bundle) analogues of a linear scheme S which operate in N and M, respectively, then* (2.5) *is valid in this geometric setting.*

Linear schemes which meet our requirements are the modified butterfly scheme and Kobbelt's interpolatory quad scheme [13]. The butterfly scheme was proposed by Dyn et al.[5]. It was modified by Zorin [31] to produce smooth limits near extraordinary vertices. An analysis of both schemes can be found in [27].

As a consequence of Corollary 2.4, the Riemannian analogues (1.2) and (1.5) of the modified butterfly scheme and of Kobbelt's interpolatory quad scheme fulfill (2.5). Other analogues meeting the requirements of the corollary are the *projection analogue* and the *geodesic analogue* analyzed in [23].

The exact value of the smoothness index ω defined by (2.4) depends on the valences of the vertices in the combinatorics K. Numerical evaluation in dependence of the valence of the vertices for Kobbelt's scheme can be found in [27].

The modified butterfly scheme has some properties which are very nice for our purposes:

Corollary 2.5. *Let T' and T'' be geometric (bundle) analogues of the modified butterfly scheme in N and M, respectively, and assume that the initial mesh which covers N fulfills* (2.1). *Then for continuous $f : N \to M$ and and any positive γ, which is smaller than the smoothness index of the butterfly scheme on regular meshes,*

$$f \in \mathrm{Lip}_\gamma(N, M) \quad \text{if and only if} \quad \|d_i\|_\infty \leq C 2^{-i\gamma}.$$

Here d_i are the coefficients of the multiscale transform (1.7).

The above corollary involves the smoothness index of the butterfly scheme on regular meshes which is known to lie in the intervall $[1.44, 2]$. The lower bound is given in [8], and the upper bound is clear since the 4-point scheme does not produce C^2 limits. Note that the statement of Corollary 2.5 does not depend on the valences of the vertices in the combinatorics K, and that the decay conditions are as in the regular mesh case.

2.2. **Proofs.** The main part of this section is devoted to the proof of Theorem 2.3. We begin by providing some information on the invariance properties of Hölder-Zygmund functions.

For an open subset $U \subset \mathbb{R}^n$ and $0 < \alpha \leq 1$ we define *the Hölder classes* $C^{1,\alpha}(U, \mathbb{R}^d)$ as the space of C^1 functions $f : U \to \mathbb{R}^d$ such that, for the differential of f, $\|d_x f(x) - d_y f\| \leq C\|x - y\|^\alpha$, for all $x, y \in U$.

We need the following properties of Hölder-Zygmund and Hölder classes which mainly concern invariance under composition and multiplication.

Proposition 2.6. *Assume that $0 < \gamma < 2$ and that $0 < \alpha \leq 1$ such that $\alpha \geq \gamma - 1$. Consider $f \in \mathrm{Lip}_\gamma(\mathbb{R}^n, \mathbb{R}^d)$. Let U, V be open sets in $\mathbb{R}^n$, and let $g : U \to V$ be a C^1 diffeomorphism with $g \in C^{1,\alpha}(U, \mathbb{R}^d)$. Furthermore, assume that U', V' are open sets in $\mathbb{R}^d$, and that $h : U' \to V'$ is a C^1 diffeomorphism with $h \in C^{1,\alpha}(U', V')$. Last but not least, let $K \subset W$ be a compact set contained in the open set W, and $f' : W \to \mathbb{R}^d$ be a continuous bounded function which fulfills $\|\Delta_h^2 f(x)\| < Ch^\gamma$ for all $x \in K$ and $\|h\| < h_0$, where $B(y, 2h_0) \subset W$ for all $y \in K$. Under the assumption that all sets are connected and contain 0, we have the following statements.*

(i) *If $u \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ with $\operatorname{supp} u \subset \operatorname{int} K$, then the product $uf' : \mathbb{R}^n \to \mathbb{R}^d$ (extended by 0 outside K) belongs to $\mathrm{Lip}_\gamma(\mathbb{R}^n, \mathbb{R}^d)$.*

(ii) *If $L \subset U$ is compact, then there is an open neighborhood N of $g(L)$, such that $g^{-1} \in C^{1,\alpha}(N, \mathbb{R}^n)$.*

(iii) *If f is compactly supported in V, then $f \circ g \in \mathrm{Lip}_\gamma(\mathbb{R}^n, \mathbb{R}^d)$. Furthermore, $\|f \circ g\|_{\mathrm{Lip}_\gamma} \leq C\|g\|_{C^{1,\alpha}(\operatorname{supp} f)}\|f\|_{\mathrm{Lip}_\gamma}$.*

(iv) *If f has compact support and $\operatorname{im} f \subset U'$, then $h \circ f \in \mathrm{Lip}_\gamma(\mathbb{R}^n, \mathbb{R}^d)$.*

Proof. Note that for $0 < \alpha < 1$ that the Hölder spaces $C^{1,\alpha}(\mathbb{R}^n)$ and the Hölder-Zygmund spaces $\mathrm{Lip}_{1+\alpha}(\mathbb{R}^n)$ coincide (which is, in general, no longer true, if we replace $\mathbb{R}^n$ by an open set U).

In order to avoid pathologies (arising from the choice of domains), the Hölder functions and the Hölder-Zygmund functions in the statements are compactly supported or defined in a neighbourhood of the open set of interest− not only on the open set itself. This allows us to use certain results for the $\mathbb{R}^n$ case rather than have to deal with problems at the boundaries of the domain. In particular, certain proofs given for the $\mathbb{R}^n$ case which are based on differences and moduli of continuity (which are quanities of a local nature) carry over to our setting.

In case $\gamma \neq 1$, (i) is a straightforward computation. For $\gamma = 1$, we can use the representation [3, Equ. (2.4)] and proceed in a way analogous to the proof of Proposition 3 in [3]. This is justified, since our setup allows to apply [3, Equ. (2.2)].

We come to (ii). The corresponding statement for the $\mathbb{R}^n$ case is stated as Theorem 2.1 in [2] and is there attributed to Norton [15]. The argumentation in [2] is a local one, and choosing N as a set with compact closure in $g(U)$ yields (ii).

For $\gamma \neq 1$, statements (iii) and (iv) in the $\mathbb{R}^n$ case are Lemma 2.2 and Lemma 2.3 of [2]. Again, by the locality of the arguments in the proof of these lemmas and the compactness of $\operatorname{supp} f$, (iii) holds true as stated.

The $\mathbb{R}^n$ statement analogous to (iii) for $\gamma = 1$ is the *composition theorem* of [16]. Its proof which is based on certain moduli of continuity also applies to the situation in (iii).

A similar statement to (iv) in the $\mathbb{R}^n$ case for $\gamma = 1$ is Theorem 2 of [3]. The difference is that only the case $d = 1$ is stated. However, the moduli η and ν employed in [3] can be generalized to arbitrary dimension d in the obvious way. Then the generalization to

arbitrary d of Proposition 4 and Theorem 6 in [3] remains valid. An analysis of the proofs of Proposition 4 and Theorem 6 of [3] shows that they also apply to the situation in (iv) (every $C^{1,\alpha}$ function fulfills the condition [3, Equ. (1.1)]). □

With the help of Proposition 2.6 we are able to show Proposition 2.2.

Proof of Proposition 2.2. It is sufficient to show the result for connected N. We assume that the conditions of Definition 2.1 are fulfilled for a function f and geodesic balls $B(x_i, r)$ and $B(y_j, R)$, respectively. We consider another such set of balls $B'(z_k, r')$ and $B'(v_l, R')$ with respect to different Riemannian metrics on N and M, respectively. Consider the partition of unity $\{\varphi_i\}$ and the functions f_i as in Definition 2.1, and an analogous partition of unity $\{\varphi'_k\}$ and the corresponding functions f'_k corresponding to the different choice of balls. We have to show that, for all k, $f'_k \in \mathrm{Lip}_\gamma(\mathbb{R}^m, \mathbb{R}^n)$.

To that end, we choose some small enough R'' and finitely many balls $B'(q_t, R'')$ which cover $f(N)$ such that, for each t, there is j and l with $B'(q_t, R'') \subset B(y_j, R)$ and $B'(q_t, R'') \subset B'(v_l, R')$. Then we choose some small enough r'' and finitely many balls $B'(p_s, r'')$ which cover N such that, for each s, there is i and k with $B'(p_s, r'') \subset B(x_i, r)$ and $B'(p_s, r'') \subset B'(z_k, r')$, and such that there is t with $f(B'(p_s, r'')) \subset B'(q_t, R'')$. We let $\{\varphi''_s\}$ be a partition of unity subordinate to the balls $B'(p_s, r'')$.

We construct the functions f''_s following Definition 2.1, using the balls $B'(p_s, r'')$, $B'(q_t, R'')$ and the partition of unity $\{\varphi''_s\}$. Proposition 2.6 yields $f''_s \in \mathrm{Lip}_\gamma(\mathbb{R}^m, \mathbb{R}^n)$ for all s.

Consider now f'_k. Modulo a change of exponential charts, we can write $f'_k = \sum_s \psi_s f''_s$ with smooth functions ψ_s with compact support. By Proposition 2.6, this change of exponential charts leaves the Lip_γ property invariant. Again by Proposition 2.6, multiplication with ψ_s leaves the Lip_γ property invariant. Thus $f'_k \in \mathrm{Lip}_\gamma(\mathbb{R}^m, \mathbb{R}^n)$. □

We introduce some notation we need for the proof of the following theorem. For a function p_n on V_n for some k-regular mesh and a subset B of V_n, we define

$$\mathcal{D}_B(p_n) = \sup\{\|p_n(v) - p_n(w)\| : v \text{ and } w \text{ are neighbors in } B\}.$$

We drop the index B, if $B = V_n$. $\mathcal{D}_B$ gives an upper bound on the coarseness of the corresponding mesh on B.

Theorem 2.10 in [25] is only concerned with C^1 smoothness. We need the following generalization of that theorem which applies to Hölder functions.

Theorem 2.7. *Let S be a linear subdivision scheme which meets the requirements of Section 1.4, and let T be in proximity with S. Let $\omega > 1$ be the smoothness index of S for a k-regular mesh. If T converges for input p_0 (which is guaranteed if p_0 is dense enough in the sense that $\mathcal{D}(p_0)$ is small) then this limit is in $C^{1,\alpha-1}$ whenever $1 < \alpha < \omega$.*

Proof. We first consider linear subdivision and then use the results for that case to obtain the corresponding statement for the nonlinear case.

We consider the limit function $h = S_{\infty,0}p_0$ for input p_0 and its restriction $h_m = h|_{D_m}$ to the ring D_m. As before, λ denotes the subdominant eigenvalue of the subdivision matrix A and μ denotes the modulus of the sub-subdominant eigenvalue(s). We are ordering the eigenvalues of A by their modulus, $1 > \lambda = \lambda > |\mu_3| \geq \ldots \geq |\mu_r| \geq \ldots$. Then h_0 can be represented as $h_0 = \sum_r \sum_{j=0}^{l_r} \beta_r^j e_r^j$ with $\{e_r^j\}$ being the eigen-rings of the subdivision scheme [18] and β_r^j being coefficients. Here the index r corresponds to the eigenvalues and the index j corresponds to the Jordan block of the corresponding eigenvalue. The limit

function on the m-th ring has the nice representation

$$h_m = \beta_0 + \beta_1\lambda^m e_1(2^m\cdot) + \beta_2\lambda^m e_2(2^m\cdot) + \sum_r \sum_{l=0}^{l_r} \binom{m}{l}\mu_r^{m-l} \sum_{i=l}^{l_r} \beta_r^i e_r^{i-l}(2^i\cdot) =: h'_m + h''_m. \tag{2.6}$$

See Chapter 4.6 of [18] for details.

Consider now the function $h_m \circ \chi^{-1}$, i.e., we look at the characteristic parametrization of the limit. By [18], the differential of h''_m as defined by (2.6) fulfills $d(h''_m \circ \chi^{-1}) = O(\lambda^{-m}(\mu s)^m)$ uniformly on D_m as $m \to \infty$ for every $s > 1$.

Assume that α is a real number with $1 < \alpha < \omega$. Since limits on regular meshes are $C^{1,\alpha-1}$, for all points x, y in, say, three consecutive rings $\chi(D_{m-1}) \cup \chi(D_m) \cup \chi(D_{m+1})$ the Hölder condition

$$\|d_x(h \circ \chi^{-1}) - d_y(h \circ \chi^{-1})\| \le C\|x - y\|^{\alpha-1} \tag{2.7}$$

is fulfilled for some constant $C > 0$ which is independent of the particular m.

We consider the situation near the central point 0. We write h' for the function defined on each D_m by h'_m ($m \in \mathbb{N}$) and by β_0 in 0 (h'_m is defined in (2.6)). Analogously, we define h'' with the difference that $h''(0) = 0$. Then $h' \circ \chi^{-1}$ is an affine-linear function and therefore $d_x(h' \circ \chi^{-1}) - d_y(h' \circ \chi^{-1}) = 0$. Hence

$$\|d_x(h \circ \chi^{-1}) - d_0(h \circ \chi^{-1})\|/\|x\|^{\alpha-1} = \|d_x(h'' \circ \chi^{-1})\|/\|x\|^{\alpha-1}.$$

Now, consider $x \in \chi(D_m)$. Two consecutive rings are λ-homothetic. So there are $k, K > 0$ which are independent of x and m such that $k\lambda^m \le \|x\| \le K\lambda^m$. Therefore, there are $C_1, C_2 > 0$ such that

$$\begin{aligned}\|d_x(h''_m \circ \chi^{-1})\|/\|x\|^{\alpha-1} &\le C_1\|d_x(h''_m \circ \chi^{-1})\|/\lambda^{m(\alpha-1)} \\ &\le C_2\lambda^{-m}(\mu s)^m/\lambda^{m(\alpha-1)} = (s(\mu/\lambda^\alpha))^m.\end{aligned}$$

We choose $s > 1$ such that $\rho = s\lambda^{\nu-\alpha} < 1$. Then $s(\mu/\lambda^\alpha) = (\mu/\lambda^\nu)(s\lambda^{\nu-\alpha}) = \rho$. This is because the first factor equals 1 by definition of ν. Then, $\|d_x(h_m \circ \chi^{-1})\|/\|x\|^{\alpha-1} \le C_2\rho^m \le C_2$. This implies that the Hölder condition (2.7) holds also in 0.

For points x and y, which lie in two rings, say $\chi(D_r)$ and $\chi(D_s)$, with $|r - s| > 2$, we estimate differentials by

$$\|d_x(h \circ \chi^{-1}) - d_y(h \circ \chi^{-1})\| \le \|d_x(h \circ \chi^{-1}) - d_0(h \circ \chi^{-1})\| + \|d_y(h \circ \chi^{-1}) - d_0(h \circ \chi^{-1})\|.$$

By the contraction of the rings, $\|x - y\|^\alpha \ge c\max(\|x\|^\alpha, \|y\|^\alpha)$ for some $c > 0$ which is independent of x and y as long as $|r - s| > 2$. This yields a (larger) constant C' such that (2.7) still holds with C replaced by C'. Altogether, this implies that the limit of linear subdivision is a $C^{1,\alpha-1}$ function.

Since we now know that S produces $C^{1,\alpha-1}$ limits for $\alpha < \omega$, we can base the proof for the nonlinear case upon the perturbation arguments used in the proof of Theorem 2.10 of [25]. We assume $\alpha < \omega$. We point out where modifications are necessary. First of all, note that for a function u on $\mathbb{R}^n$ and some $h > 0$, we have $c, C > 0$ such that the dilated function $u(h\cdot)$ can be estimated by $ch^\alpha\|u\|_{C^{1,\alpha-1}} \le \|u(h\cdot)\|_{C^{1,\alpha-1}} \le Ch^\alpha\|u\|_{C^{1,\alpha-1}}$. ($C$ is a generic constant, which can change from line to line from now on.) With this in mind, we can use

the argumentation of Proposition 2.12 in [25] to obtain that

$$\begin{aligned}\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0)\circ\chi^{-1}|_{\chi(D_n)}\|_{C^{1,\alpha-1}} \\ \leq C(2^{i-n}\lambda^{-n})^{\alpha}\|(T_i - S_i)T_{i-1,0}p_0|_{\mathrm{ctrl}^{i+1}(D_n)}\|_{\infty}.\end{aligned}$$

Invoking this estimate yields a statement analogous to [25, Equ. (210)] for the rings near the extraordinary vertex:

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0)\circ\chi^{-1}|_{\chi(D_n)}\|_{C^{1,\alpha-1}} \leq C\gamma^{(2-\alpha)i}\mathcal{D}_{\mathrm{ctrl}^0(D')}(p_0)^2, \tag{2.8}$$

where $\gamma := \max(2^{-1},\lambda)$. The $C^{1,\alpha-1}$ version of (2.12) in [25] reads

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0)\circ\chi^{-1}|_{\chi\left(\cup_{m=i}^{\infty}D_m\cup\{0\}\right)}\|_{C^{1,\alpha-1}} \leq C\lambda^{(2-\alpha)i}\mathcal{D}_{\mathrm{ctrl}^0(D')}(p_0)^2. \tag{2.9}$$

The estimates (2.8) and (2.9) now imply that the limit using T is $C^{1,\alpha-1}$. This follows with minor modifications from the proofs of Proposition 2.13 and Theorem 2.10 of [25]. □

The next proposition treats vector space data defined over a 2-manifold. It is a special case of our main result.

Proposition 2.8. *Let the interpolatory scheme T' act on the smooth compact 2-manifold N and assume that it is in proximity to linear interpolatory scheme S. Assume that the initial mesh (K_0,p_0) in N fulfills the non-degeneracy property 2.1. Let ω be the smoothness index of S for that mesh. We consider the linear version of the transform (1.7) and a continuous function $f: N\to\mathbb{R}^d$. Then for any γ with $0<\gamma<\omega$ we have the characterization*

$$f\in \mathrm{Lip}_\gamma(N,\mathbb{R}^d) \quad \text{if and only if} \quad \sup_{i\in\mathbb{N}_0}\|d_i\|_{i,\gamma}\leq C. \tag{2.10}$$

Furthermore, $\|f_0\|_\infty + \sup_{i\in\mathbb{N}_0}\|d_i\|_{i,\gamma}$ provides an equivalent norm on $\mathrm{Lip}_\gamma(N,\mathbb{R}^d)$.

Proof. The proof of this statement takes some time. So we split it into several parts. Part (1) reduces the statement to a statement involving only one extraordinary vertex. In parts (2)–(5) we show the reduced statement: Part (2) is the 'only if'-part in case $\gamma\neq 1$. The 'if'-part of the statement is treated in part (3). In part (4) we explain why $\|f_0\|_\infty+\sup_{i\in\mathbb{N}_0}\|d_i\|_{i,\gamma}$ defines an equivalent norm on $\mathrm{Lip}_\gamma(N,\mathbb{R}^d)$ in case $\gamma\neq 1$. In Part (5) we show the 'only if'-part of the statement and treat the norm equivalence for $\gamma=1$.

We need the sets V_i and X_i which were defined in Section 1.3 and at the beginning of Section 1.2, respectively. The subsets V_i^j and X_i^j are given by (2.2) and the lines following (2.2), respectively. We let C be a generic constant which can change from line to line.

(1) We reduce the statement to a more accessible situation near extraordinary vertices. To that purpose, consider the neighborhood of an extraordinary vertex $x\in X_0\subset N$ and the corresponding point $0\in V_0$ in the glued domain D. Denote by $\bar{X}_i=\chi(V_i)$ the image of V_i under characteristic parametrization. With the diffeomorphism κ of (2.1), $\chi\circ\kappa^{-1}$ is a local diffeomorphism mapping x to $0\in\mathbb{R}^2$. Thus $\chi\circ\kappa^{-1}$ sends neighbors of $x\in X_i\subset X_0$ to neighbors of $0\in\bar{X}_i$. For a visualization see Figure 3.

Now choose finitely many small geodesic balls $B(y_j,r)$ which cover N, such that each $\kappa^{-1}(B(y_j,r))$ is completely contained in some characteristic chart neighborhood. Let $\{\psi_j\}$ be C^∞ functions such that each ψ_j is supported in $B(y_j,r)$ and equal to 1 on $B(y_j,r-\varepsilon)$, where $\varepsilon>0$ is so small such that the balls $B(y_j,r-\varepsilon)$ still cover N. If $f\in\mathrm{Lip}_\gamma(N,\mathbb{R}^d)$ then $f\psi_j$ is compactly supported in $B(y_i,r)$ and the extension of its chart representation with 0 outside the ball is in $\mathrm{Lip}_\gamma(\mathbb{R}^2,\mathbb{R}^d)$. Let us denote this extension also by $f\psi_j$.

The mapping $u_j = \chi \circ \kappa^{-1} \circ \exp_{y_j}^{-1}$ is a diffeomorphism from $B(0,r)$ into $\mathbb{R}^2$. Its image contains the compact set $u_j(\operatorname{supp} f\psi_j)$. By Theorem 2.7 the inverse u_j^{-1} is $C^{1,\alpha}$ for all $\alpha < \omega$. Therefore, Proposition 2.6 implies that $(f\psi_j) \circ u_j^{-1} \in \operatorname{Lip}_\gamma(\mathbb{R}^2, \mathbb{R}^d)$ (with the usual 0-extension). This means that a Hölder-Zygmund function on N transforms to a Hölder-Zygmund function near 0 in the image of a characteristic chart.

Conversely, if we have a Hölder-Zygmund function g in the image of a characteristic chart which is compactly supported in $u_j(B(0,r))$, we use Proposition 2.6 to obtain that $g \circ u_j$ is Hölder-Zygmund on N (with extension by 0). For a Hölder-Zygmund function g defined on $\chi(D)$ which is not necessarily supported in $u_j(B(0,r))$ we can multiply g with $\psi \circ u_j^{-1}$ to obtain a function that has support in $u_j(B(0,r))$ and apply the above to obtain a Hölder-Zygmund function on N.

We define the details $\bar{d}_i$ and the control sets $\bar{X}_i^j$ analogous to the details d_i and the control sets X_i^j, only by replacing $X_i \subset N$ by $X_i' \subset \mathbb{R}^2$. Then, locally near an extraordinary vertex, the details d_i of f given on N and the details $\bar{d}_i$ of $f \circ \kappa \circ \chi^{-1}$ are equal.

If a ball $B(y_j, r)$ in N does not contain an extraordinary vertex, then we are in the regular mesh case. But this is a special instance of a 4-regular mesh in case of quad meshes, and a 6-regular mesh in case of triangular meshes which is treated by the general k-regular case.

Summing up, it is enough to show the following reduced statement for the k-regular mesh for a continuous function f with compact support in a neighborhood of $\chi(D')$:

$$f \in \operatorname{Lip}_\gamma(\chi(D), \mathbb{R}^d) \quad \text{if and only if} \quad \sup_{i\in\mathbb{N}} \|f_i - S_{i-1}f_{i-1}\|_{i,\gamma} \le C. \tag{2.11}$$

We also show that $\|f_0\|_\infty + \sup_{i\in\mathbb{N}_0} \|f_i - S_{i-1}f_{i-1}\|_{i,\gamma}$ provides an equivalent norm on

$$\operatorname{Lip}_\gamma^K(\chi(D), \mathbb{R}^d) = \{f \in \operatorname{Lip}_\gamma(\chi(D), \mathbb{R}^d) : \operatorname{supp} f \subset K\} \tag{2.12}$$

for some fixed but arbitrary neighborhood K of 0. Then the corresponding statement in the proposition follows from Proposition 2.6(iii).

For the further proof we let $d = 1$, since the right hand expression in (2.11) is equivalent (lower and upper constants) to the maximum of the corresponding component-wise expressions.

(2) We show the 'only if'-part of (2.11) for $\gamma \neq 1$. So our assumption is that $f \in \operatorname{Lip}_\gamma(\chi(D), \mathbb{R})$. f_i denotes the restriction of f to $\bar{X}_i$. The subdivision scheme S acts on functions on V_i as linear operator S_i and thus also on functions on $\bar{X}_i$. We denote this operator on functions on $\bar{X}_i$ by S_i, too. We abuse notation and also use $S_{\infty,i}$ to denote the operator which maps input on $\bar{X}_i$ to its limit w.r.t. characteristic parametrization.

Consider the restriction of f_i to the sets $\bar{X}_i^j$ (the index i corresponds to level i and the index j to the ring j near an irregular vertex). In the course of the proof we have to estimate the norm of $(f_i - S_{i-1}f_{i-1})|_{\bar{X}_i^j}$. We have to distinguish two cases depending on whether $l := i - j$, i.e., the difference between level and ring index, is small or not.

If we choose l sufficiently large, say $l \ge l_0$, we get that

$$\|(f_i - S_{i-1}f_{i-1})|_{\bar{X}_i^j}\|_\infty \le \|(f - S_{\infty,i-1}f_{i-1})|_{\chi(D_j'')}\|_\infty, \tag{2.13}$$

where we set $D_j'' = D_{j-1} \cup D_j \cup D_{j+1}$. This is a consequence of S being interpolatory and the fact that the contol sets $\bar{X}_i^j$ on level i of $\chi(D_j)$ are contained in D_j''.

For $l = i-j < l_0$, we find $r \in \mathbb{N}$ such that $\bar{X}_i^j \subset \chi(D'_{j-r})$, where $D'_l := D' \backslash (D_0 \cup \ldots \cup D_{l-1})$. ($D'$ was defined as the union of all the rings D_i, $i \in \mathbb{N}$, and 0 in Section 1.3.) Then

$$\|(f_i - S_{i-1}f_{i-1})|_{\bar{X}_i^j}\|_\infty \leq \|(f - S_{\infty,i-1}f_{i-1})|_{\chi(D'_{j-r})}\|_\infty. \tag{2.14}$$

Observe that showing

$$\|(f - S_{\infty,i-1}f_{i-1})|_{\chi(D_j)}\|_\infty \leq C\lambda^{j\gamma}2^{(j-i)\gamma} \quad \text{and} \tag{2.15}$$

$$\|(f - S_{\infty,i-1}f_{i-1})|_{\chi(D'_i)}\|_\infty \leq C\lambda^{i\gamma} \tag{2.16}$$

is enough to complete this part of the proof. This is because (2.15) and (2.16) together imply that (2.15) is valid with D_j replaced by D''_j or by D'_i, respectively, if we enlarge the constant C. Then (2.13) and (2.14) imply $\|(f_i - S_{i-1}f_{i-1})|_{\bar{X}_i^j}\|_\infty \leq C\lambda^{j\gamma}2^{(j-i)\gamma}$, where C is independent of i and j. This is the right-hand side of (2.11).

We show the approximation estimates (2.15) and (2.16). If $\gamma > 1$, we write $f = f(v) + d_v f(\cdot - v) + g(\cdot)$ with $g(x) = O(\|x - v\|^\gamma)$ for $x \to v$ by our assumption. The linear bounded operator which first samples f and then maps the result to the limit of subdivision reproduces constants and linear functions w.r.t. the characteristic parametrization. So, for a vertex $v \in \bar{X}_{i-1}$, we have $S_{\infty,i-1}f_{i-1} = f(v) + d_v f(\cdot - v) + h(\cdot)$ for some h with $h(x) = O(\|x - v\|^\gamma)$. Then, if v is a point in $\bar{X}_{i-1}$ nearest to x, we obtain

$$f(x) - S_{\infty,i-1}f_{i-1}(x) = g(x) - h(x) = O(\|x - v\|^\gamma) \text{ for } x \to v. \tag{2.17}$$

If $\gamma < 1$, the estimate (2.17) is shown in the same way, without using differentials.

In order to estimate $\|x-v\|$ in (2.17) we introduce the notation $\sigma(A, B) = \sup_{x\in A} \inf_{v\in B}\|x-v\|$. By the definition of V_k, $\sigma(D'_k, V_k^k) = O(2^k)$ and $\sigma(D_r, V_k^r) = O(2^k)$ as $k \to \infty$ uniformly in r for $r < k$. Because the charactistic map is a diffeomorphism on each ring D_k fulfilling the scaling relation $\chi(2\cdot) = \lambda\chi$, we have that $\sigma(\chi(D'_k), \bar{X}_k^k) = O(\lambda^k)$ and that $\sigma(\chi(D_r), \bar{X}_k^r) = O(\lambda^r 2^{r-k})$ as $k \to \infty$ uniformly in r for $r < k$.

If we apply this to (2.17) for $x \in D'_{i-1}$, we get $\inf_{v\in\bar{X}_{i-1}} \|x - v\| = O(\lambda^i)$. Also, for $j \geq i - 1$, and $x \in D_j$, we obtain that $\inf_{v\in\bar{X}_{i-1}} \|x - v\| = O(\lambda^i 2^{i-j})$.

Then plugging $\|x-v\| \leq C\lambda^j 2^{(j-i)}$ into (2.17) and enlarging the constant C yields (2.15) and (2.16). This completes part (2) of the proof.

(3) We show the 'if'-part of (2.11). The continuous functions $g_i = S_{\infty,i}f_i$ uniformly convergence to f on $\chi(D)$ for the following reason: Since S is interpolatory, for a vertex $v \in \bar{X}_i$ nearest to x, we get that

$$\begin{aligned}\|g_i(x) - f(x)\| &\leq \|S_{\infty,i}f_i(x) - S_{\infty,i}f_i(v)\| + \|f(x) - f(v)\| \\ &\leq C \sup\{\|f(v) - f(w))\| : v, w \text{ neighboring vertices}\} + \|f(x) - f(v)\|,\end{aligned}$$

and the right hand side tends to 0 as $i \to \infty$.

The right-hand side of (2.11) implies that, for $i > j$,

$$\|g_i - g_{i-1}|_{\chi(D_j)}\|_\infty \leq \|S_{\infty,0}\|\|f_i - Sf_{i-1}|_{\bar{X}_i^j}\|_\infty \leq C'\|S_{\infty,0}\|2^{(j-i)\gamma}\lambda^{j\gamma}. \tag{2.18}$$

Here C' is the constant in the decay condition which depends on f. In this part, we continue to use the symbol C as a generic constant which can change from term to term, but we only employ it if it does not depend on f. We use (2.18) to quantify the distance between

f and the approximants g_i on the ring $\chi(D_j)$:

$$\begin{aligned}\|f-g_i|_{\chi(D_j)}\|_\infty &\le \sum_{k=i+1}^{\infty}\|g_k-g_{k-1}|_{\chi(D_j)}\|_\infty\\ &\le C'\|S_{\infty,0}\|\sum_{k=i+1}^{\infty}2^{(j-k)\gamma}\lambda^{j\gamma}\le C'C2^{(j-i)\gamma}\lambda^{j\gamma}.\end{aligned}\tag{2.19}$$

We consider the inner domains $\chi(D'_j)$ now. Using the right-hand side of (2.11), an estimate analogous to (2.18) yields $\|g_i-g_{i-1}|_{\chi(D'_j)}\|_\infty \le C'C\lambda^{i\gamma}$, whenever $i\le j$. Then,

$$\begin{aligned}\|f-g_i|_{\chi(D'_j)}\|_\infty &\le \sum_{k=i+1}^{\infty}\|g_k-g_{k-1}|_{\chi(D_j)}\|_\infty\\ &\le C\|S_{\infty,0}\|(\sum_{k=i+1}^{j}\lambda^{k\gamma}+\sum_{k=j+1}^{\infty}2^{(j-k)\gamma}\lambda^{j\gamma})\\ &\le C'C\lambda^{i\gamma}\sum_{k=1}^{\infty}\max(2^{-1},\lambda)^{k\gamma}\le C'C\lambda^{i\gamma}.\end{aligned}\tag{2.20}$$

We proceed to estimate second differences, beginning on the rings $\chi(D_j)$. By enlarging the constant C in (2.19), the statement of (2.19) remains valid for sufficiently small ϵ-neighborhoods U_j of $\chi(D_j)$. We choose the neighborhoods U_j of $\chi(D_j)$ in such a way that each U_j is a scaled copy of the neighborhood of U_0 with factor $j\lambda$. Then there is $h_0>0$ such that, for any j, all $x\in\chi(D_j)$, and all t with $\|t\|<\lambda^j h_0$, the second difference $\Delta_t^2 f(x)$ only depends on $f|U_j$.

We let α be a real number with $\gamma<\alpha<\omega$. Consider the modulus of continuity $\omega_2^j(h,f):=\sup_{\|t\|<h}\|(\Delta_t^2 f)|_{\chi(D_j)}\|$, for $h<\lambda^j h_0$. We have the estimate

$$\begin{aligned}\omega_2^j(h,f)&\le \omega_2^j(h,f-g_n)+\sum_{i=0}^{n-1}\omega_2^j(h,g_{i+1}-g_i)+\omega_2^j(h,g_0)\\ &\le 4\|(f-g_n)|_{\chi(U_j)}\|_\infty+\sum_{i=0}^{n-1}h^\alpha\|g_{i+1}-g_i\|_{\alpha,j}+\omega_2^j(h,g_0),\end{aligned}\tag{2.21}$$

where $\|\cdot\|_{\alpha,j}:=\sup_h h^{-\alpha}\omega_2^j(h,\cdot)+\|\cdot|_{\chi(U_j)}\|_\infty$.

With the help of (2.19) and (2.20) we can estimate the first summand on the right-hand side of (2.21) by $\|(f-g_n)|_{\chi(U_j)}\|_\infty\le CC'\lambda^{\min(n,j)\gamma}2^{-\max(n-j,0)\gamma}$.

We consider the sum in (2.21). By the locality of the subdivision scheme S, the limit function locally is a linear combination of finitely many generating functions. Furthermore, on a regular mesh, an integer shift of those generating functions is a generating system for the shifted functions. Near 0 in a k-regular mesh we also have a finite generating system. Since going to a finer resolution only dilates the generating systems, we get $\|g_{i+1}-g_i\|_{\alpha,j}\le C2^{(i-j)\alpha}\lambda^{-j\alpha}\|g_{i+1}-g_i|_{\chi(U_j)}\|_\infty$ in case that $i>j$. If $i\le j$, we obtain $\|g_{i+1}-g_i\|_{\alpha,j}\le C\lambda^{-i\alpha}\|g_{i+1}-g_i|_{\chi(U_j\cup D'_j)}\|_\infty$. By combining these estimates, we get

$$\begin{aligned}\sum_{i=0}^{n-1}h^\alpha\|g_{i+1}-g_i\|_{\alpha,j}&\le C\sum_{i=0}^{n-1}h^\alpha\lambda^{-\min(i,j)\alpha}2^{\max(i-j,0)\alpha}\|g_{i+1}-g_i|_{\chi(U_j)}\|_\infty\\ &\le CC'\sum_{i=0}^{n-1}h^\alpha\lambda^{-\min(i,j)(\alpha-\gamma)}2^{\max(i-j,0)(\alpha-\gamma)}.\end{aligned}\tag{2.22}$$

We further discuss this upper bound. We consider n with $n>j$ and set $h=2^{j-n}\lambda^j$. Then,

$$\begin{aligned}h^{-\gamma}\sum_{i=0}^{n-1}h^\alpha\lambda^{-\min(i,j)(\alpha-\gamma)}2^{\max(i-j,0)(\alpha-\gamma)}&=\sum_{i=0}^{n-1}\lambda^{(j-\min(i,j))(\alpha-\gamma)}2^{(\max(i-j,0)+j-n)(\alpha-\gamma)}\\ &=\sum_{i=0}^{j-1}\lambda^{(j-i)(\alpha-\gamma)}+\sum_{i=j}^{n-1}2^{(i-n)(\alpha-\gamma)}\le C,\end{aligned}\tag{2.23}$$

where C is independent of n and j. We plug (2.23) into (2.22) and the result into (2.21). For $h = 2^{j-n}\lambda^j$ and $j < n$ we obtain

$$h^{-\gamma}\omega_2^j(h,f) \leq 4C'C + CC' + CC', \tag{2.24}$$

where the constants do not depend on j and n. Since the sequence $h_n = 2^{j-n}\lambda^j$ goes nicely to 0, it follows that there is h_0' with $0 < h_0' < h_0$ such that, for all j and h with $0 < h < h_0'\lambda^j$,

$$h^{-\gamma}\omega_2^j(h,f) \leq C'C. \tag{2.25}$$

We consider the neighborhood of the central point. Instead of D_j we consider the central domain D_j', and employ the second modulus of continuity $\tilde{\omega}_2^j(h,f) := \sup_{\|t\|<h} \|(\Delta_t^2 f)|_{\chi(D_j')}\|$, for $h < \lambda^j h_0$. Analogous to (2.21) we estimate

$$\begin{aligned}\tilde{\omega}_2^j(h,f) &\leq \tilde{\omega}_2^j(h,f-g_j) + \sum\nolimits_{i=0}^{j-1} \tilde{\omega}_2^j(h,g_{i+1}-g_i) + \tilde{\omega}_2^j(h,g_0)\\ &\leq 4\|(f-g_j)|_{\chi(U_j')}\|_\infty + \sum\nolimits_{i=0}^{j-1} h^\alpha \|g_{i+1}-g_i\|_{\alpha,j} + \omega_2^j(h,g_0),\end{aligned} \tag{2.26}$$

where the above definition of $\|\cdot\|_{\alpha,j}$ is modified by replacing U_j by U_j'. By (2.20), the first summand on the right-hand side of (2.21) is bounded from above by $\|(f-g_j)|_{\chi(U_j)}\|_\infty \leq CC'\lambda^{j\gamma}$. Similar to (2.22) and (2.23), letting $h = (c\lambda)^j$, for some c with $0 < c < 1$, which is small enough to guarantee the definedness of $\tilde{\omega}_2^j$,

$$\begin{aligned}h^{-\gamma}\sum\nolimits_{i=0}^{j-1} h^\alpha \|g_{i+1}-g_i\|_{\alpha,j} &\leq Ch^{-\gamma}\sum\nolimits_{i=0}^{j-1} h^\alpha \lambda^{-i\alpha}\|g_{i+1}-g_i|_{\chi(U_j')}\|_\infty\\ &\leq CC'h^{-\gamma}\sum\nolimits_{i=0}^{j-1} h^\alpha\lambda^{-i(\alpha-\gamma)}.\\ &= CC'\sum\nolimits_{i=0}^{j-1}\lambda^{(j-i)(\alpha-\gamma)} \leq CC',\end{aligned} \tag{2.27}$$

where the constants are independent of j. Combining these two estimates and plugging them into (2.26), we get, on the inner domain $\chi(D_j')$,

$$h^{-\gamma}\tilde{\omega}_2^j((c\lambda)^j,f) \leq C'C, \tag{2.28}$$

uniformly in j. Firstly, this yields the decay condition $\|\Delta_t^2 f(0)\| \leq C'C\|t\|^\gamma$ in the central point. Furthermore, if we consider some x in the j-th ring $\chi(D_j)$, and some y in the i-th ring with $j-i \geq 2$ then (2.28) ensures that $\|f(x) - 2f(\frac{x+y}{2}) + f(y)\| \leq CC'\|x-y\|^\gamma$. If the distance is smaller, then (2.25) applies. In summary, this shows that $f \in \mathrm{Lip}_\gamma(\chi(D),\mathbb{R}^d)$.

(4) We explain why $\|f\|_\gamma' = \|f_0\|_\infty + \sup_{i\in\mathbb{N}_0}\|f_i - S_{i-1}f_{i-1}\|_{i,\gamma}$ is an equivalent norm on $\mathrm{Lip}_\gamma^K(\chi(D),\mathbb{R}^d)$ (defined by (2.12)) for the case that $\gamma \neq 1$. By (2) and (3) the subspace of continuous functions where $\|\cdot\|_\gamma' < \infty$ coincides with Lip_γ^K. It is a straightforward computation that $\|\cdot\|_\gamma'$ defines a norm. The constants C occuring in (3) do not depend on f (for constants depending on f, we used the symbol C'). This yields that there is $C > 0$, independent of f, such that

$$\|f\|_{\mathrm{Lip}_\gamma^K} \leq C\|f\|_\gamma'. \tag{2.29}$$

Since part (3) includes the case $\gamma = 1$, (2.29) is also valid for $\gamma = 1$.

For the converse part, we have to analyze the proof of part (2). In the beginning of part (2), we reduce the statement of part (2) to (2.15) and (2.16). Examining this reduction we see that the occuring constants do not depend on f. It remains to analyze the constants occuring in the proof of (2.15) and (2.16): By careful examination, it turns out that f only

influences constants via the $O()$-term in (2.17). This means that we have to look at the Hölder constants of the functions g and h occuring in part (2). By definition, those Hölder constants are bounded by some multiple of the Hölder norm of f. Summing up, there is $C > 0$, independent of f such that

$$\|f\|'_\gamma \leq C\|f\|_{\mathrm{Lip}^K_\gamma}, \tag{2.30}$$

in case $\gamma \neq 1$. Thus those norms are equivalent for $\gamma \neq 1$. The inequality (2.30) for the case $\gamma = 1$ is shown at the end of part (5).

(5) It remains to show the 'only if'-part of (2.11) for $\gamma = 1$. To that purpose, we use interpolation theory. We refer to [17] for a thorough treatment in connection with Hölder-Zygmund classes. It is well known that Lip_1 is the interpolation space $[\mathrm{Lip}_{1-\epsilon}, \mathrm{Lip}_{1+\epsilon}]_{1/2}$.

This notation means the following: For two Banach spaces X and Y with $Y \subset X$, $[X, Y]_\theta$ is the space of all $f \in X$ such that Peetre's K-functional $K(f,t) \leq Ct^\theta$, for $0 < t \leq 1$, where

$$K(f,t) = \inf_{g\in Y} \|f - g\|_X + t\|g\|_Y.$$

The interpolation space becomes a Banach space with norm $\|\cdot\| = \sup_t t^{-\theta}K(\cdot,t)$.

We proceed in the following way: We assume that $f \in \mathrm{Lip}_1 \subset \mathrm{Lip}_{1-\varepsilon}$. Then for every t with $0 < t \leq 1$ there is $g_t \in \mathrm{Lip}_{1+\varepsilon}$ such that $t^{-1/2}\|f - g_t\|_{\mathrm{Lip}(1-\varepsilon)} + t^{1/2}\|g_t\|_{\mathrm{Lip}(1+\varepsilon)} < C$, where C does not depend on t.

We let $h_t = f - g_t$. We consider the coefficients under the multiscale transform of f, h_t and g_t on $\bar{X}_i^j$. We denote these coefficients on $\bar{X}_i^j$ by $d(f), d(g_i), \ldots$. By (4) we have $\|d(h_t)\|_\infty < C'2^{(j-i)(1-\varepsilon)}\lambda^{j(1-\varepsilon)}\|h_t\|_{\mathrm{Lip}(1-\varepsilon)}$, and $\|d(g_t)\|_\infty < C''2^{(j-i)(1+\varepsilon)}\lambda^{j(1+\varepsilon)}\|g_t\|_{\mathrm{Lip}(1+\varepsilon)}$,

By applying the triangle inequality and letting $t^{1/2} = 2^{(j-i)\varepsilon}\lambda^{j\varepsilon}$ we get

$$\begin{aligned}
2^{i-j}\lambda^{-j}\|d(f)\|_\infty &\leq 2^{i-j}\lambda^{-j}\|d(h_t)\|_\infty + 2^{i-j}\lambda^{-j}\|d(g_t)\|_\infty \\
&\leq C'2^{-(j-i)\varepsilon}\lambda^{-j\varepsilon}\|h_t\|_{\mathrm{Lip}(1-\varepsilon)} + C''2^{(j-i)\varepsilon}\lambda^{j\varepsilon}\|g_t\|_{\mathrm{Lip}(1+\varepsilon)} \\
&\leq \max(C', C'')(t^{-1/2}\|f - g_t\|_{\mathrm{Lip}(1-\varepsilon)} + t^{1/2}\|g_t\|_{\mathrm{Lip}(1+\varepsilon)}) \\
&\leq 2C''' \max(C', C'')\|f\|_{\mathrm{Lip}_1}.
\end{aligned} \tag{2.31}$$

For the last inequality we have used the equivalence of the norm induced by the K-functional and the norm induced by second differences. (2.31) means that we have the desired decay of the multiscale coefficients if $f \in \mathrm{Lip}_1$.

Furthermore, the coefficient based norm $\|\|'_1$ from part (4) obeys

$$\|f\|'_1 \leq C\|f\|_{\mathrm{Lip}_1}.$$

The other direction, i.e., $\|f\|_{\mathrm{Lip}_1} \leq C\|f\|'_1$, was already established in (2.29). Hence $\|f\|'_1$ is an equivalent norm on Lip_1.

□

Having collected all this information we are now able to prove Theorem 2.3.

Proof of Theorem 2.3. This proof is quite long which is the reason why we split it into several parts. In part (1) we reduce the statement to a statement only involving one extraordinary vertex. We proceed similar to the proof of Proposition 2.8 which is the reason for being quite brief in this part. Part (2) is the 'only if'-part of the reduced statement, and part (3) is its 'if'-part (which is actually the hard estimate).

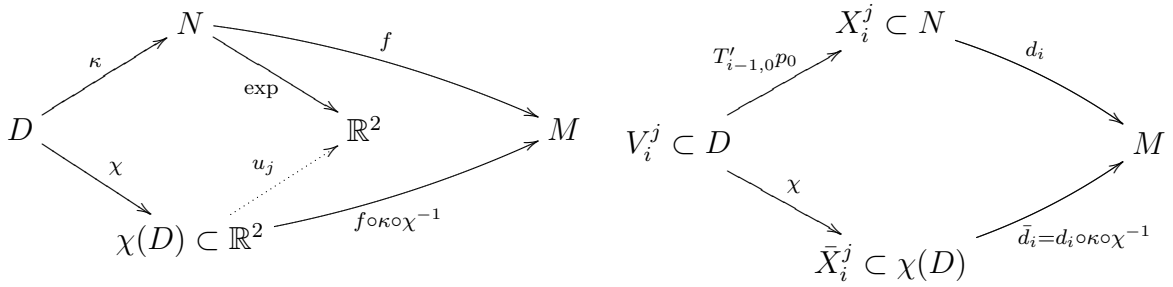

FIGURE 3. Setup for the proofs of Proposition 2.8 and Theorem 2.3.

We use the notation of the proof of Proposition 2.8. Furthermore, we use the symbol C for a generic constant which can change from line to line.

(1) Similar to the proof of Proposition 2.8 we reduce the statement to the situation near an extraordinary vertex. We show that a certain way of 'applying charts' does neither affect the Hölder-Zygmund classes nor the decay of detail coefficients.

We cover $f(N)$ with balls $B(z_k, R)$, and N with balls $B(y_j, r)$ such that each $f(B(z_k, r))$ is completely contained in one of the $B(y_j, r)$ and such that the image of each $B(z_k, r)$ under κ^{-1} is completely contained in some characteristic chart neighborhood.

We let ψ_j be C^∞ functions supported in $B(y_j, r)$ and equal to 1 in $B(y_j, r-\varepsilon)$, where $\varepsilon > 0$ is so small that the balls $B(y_j, r-\varepsilon)$ still cover N. Then the extension by 0 of $g_j = \exp_{z_k}^{-1} \circ (f\psi_j) \circ \exp_{y_j}^{-1}$ is in $\mathrm{Lip}_\gamma(\mathbb{R}^2, \mathbb{R}^d)$. Except for applying charts, g_j agrees with f on $B(y_j, r-\varepsilon)$.

With the mapping $u_j = \chi \circ \kappa^{-1} \circ \exp_{y_j}^{-1}$ already defined in part (1) of the proof of Proposition 2.8 we obtain that the 0-extension of $g_j \circ u_j^{-1}$ is in $\mathrm{Lip}_\gamma(\mathbb{R}^2, \mathbb{R}^d)$, by Proposition 2.6.

Conversely, assume that we have Hölder-Zygmund functions g_j (of order γ) such that each g_j is supported in a neighborhood of $\chi(D')$, maps to M, and agrees with $f \circ u_j^{-1}$ on $u_j(\operatorname{supp}\psi)$. Then we restrict g_j to $u_j(B(y_j, r))$, go to charts, and multiply the result with $\psi \circ u_j^{-1}$ to obtain a Hölder-Zygmund function g_j' with support in $u_j(\operatorname{supp}\psi)$. Then $g_j' \circ u_j$ (extension by 0) is Hölder-Zygmund and agrees with f on $B(y_j, r-\varepsilon)$. Furthermore, the coefficients of the multiscale transform for g_j around 0 and the transform of f near the corresponding extraordinary vertex agree.

After going to charts for M, the following statement implies the theorem. For the k-regular mesh and for a continuous function f with compact support in a neighborhood of $\chi(D')$ we have

$$f \in \mathrm{Lip}_\gamma(\chi(D), \mathbb{R}^d) \quad \text{if and only if} \quad \sup_{i\in\mathbb{N}} \|f_i - T_{i-1} f_{i-1}\|_{i,\gamma} \le C. \tag{2.32}$$

There is one more thing to explain here: We let the scheme T act in a chart which allows us to write an ordinary minus sign in (2.32). The right-hand side expression in (2.32), $\|f_i - T_{i-1} f_{i-1}\|_{i,\gamma}$, which is based on the Euclidean norm, is bounded both from above and below by constants times $\|f_i \ominus T_{i-1} f_{i-1}\|_{i,\gamma}$, which is based on the smooth bundle norm. This is a consequence of the properties of $\ominus$, the smoothness of the bundle norm and the compactness of the image of f. For details, see [7].

(2) We show the 'only if'-part of (2.32), assuming $f \in \mathrm{Lip}_\gamma(\chi(D), \mathbb{R}^d)$. Equation (2.11) yields that $\sup_{i\in\mathbb{N}_0} \|f_i - S_{i-1} f_{i-1}\|_{i,\gamma} \le C'$. We consider the sets $\bar{X}_i^j$ and observe $\|f_i -$

$S_{i-1}f_{i-1}|_{\bar{X}_i^j}\|_\infty \leq C'2^{(j-i)\gamma}\lambda^{\gamma^j}$. Since S and T fulfill the proximity condition (1.10),

$$\begin{aligned}\|f_i - T_{i-1}f_{i-1}|_{\bar{X}_i^j}\|_\infty &\leq \|f_i - S_{i-1}f_{i-1}|_{\bar{X}_i^j}\|_\infty + \|(S_{i-1}f_{i-1} - T_{i-1}f_{i-1})|_{\bar{X}_i^j}\|_\infty \\ &\leq C'2^{(j-i)\gamma}\lambda^{\gamma^j} + C\mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1})^2. \end{aligned} \tag{2.33}$$

Here we let $\tilde{X}_i^j = \bar{X}_i^j$, if $i \geq j$, and $\tilde{X}_i^j = \bar{X}_i^i$, if $i < j$. Then $\mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1})^2$ is the difference of function values of f on neighboring vertices in $\tilde{X}_{i-1}^j$. Since we consider $f \in \mathrm{Lip}_\gamma$ over the characteristic parametrization, neighboring vertices in $\tilde{X}_{i-1}^j$ have distance $2^{-\min(i-j-1,0)}\lambda^{\min(j,i-1)}$.

If $f \in \mathrm{Lip}_\gamma$, then $f \in \mathrm{Lip}_{\gamma/2+\varepsilon}$, when we choose ε such that $0 < \varepsilon < \max(1-\gamma/2, \gamma/2)$. This choice of ε guarantees that $\gamma/2 + \varepsilon < 1$. The the Lipschitz norm based on first differences is an equivalent norm on $\mathrm{Lip}_{\gamma/2+\varepsilon}$. Hence, since $f \in \mathrm{Lip}_{\gamma/2+\varepsilon}$, and all f_i are samples of f,

$$\mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1}) \leq 2^{-\min(i-j-1,0)(\gamma/2+\varepsilon)}\lambda^{\min(j,i-1)(\gamma/2+\varepsilon)}. \tag{2.34}$$

Plugging (2.34) into (2.33) yields the decay of the multiscale transform w.r.t T which is required by (2.32).

(3) We consider now the 'if'-part of (2.32), i.e., we assume a continuous function f having the decay as stated by (2.32). We again look at the control sets $\tilde{X}_i^j$. By assumption, the decay conditions for $i > j$ read:

$$\|(f_i - T_{i-1}f_{i-1})|_{\tilde{X}_i^j}\| \leq C_f 2^{-(i-j)\gamma}\lambda^{j\gamma}. \tag{2.35}$$

If $i \leq j$, then the right hand side is replaced by $C_f\lambda^{i\gamma}$. Here C_f is a constant which depends on the continuous function f, but is neither dependent on the 'ring-index' j nor on the detail level i. Our aim is to show that (2.32) and the corresponding statement for $i \leq j$ imply that for $i > j$,

$$\|(f_i - S_{i-1}f_{i-1})|_{\tilde{X}_i^j}\| \leq C'2^{-(i-j)\gamma}\lambda^{j\gamma}, \tag{2.36}$$

and for $i \leq j$, the same with the right hand side replaced by $C'\lambda^{i\gamma}$. Here the constant C' should not depend on i or j. Once (2.36) is proved we apply (2.11), and obtain that $f \in \mathrm{Lip}_\gamma$ as desired.

It remains to show (2.36) which will take some time. We start by invoking the proximity and decay conditions to obtain the following estimate for $i > j$:

$$\begin{aligned}\|(f_{i+1} - S_if_i)|_{\tilde{X}_{i+1}^j}\| &\leq \|(f_{i+1} - T_if_i)|_{\tilde{X}_{i+1}^j}\| + \|(T_if_i - S_if_i)|_{\tilde{X}_{i+1}^j}\| \\ &\leq C_f 2^{-(i+1-j)\gamma}\lambda^{j\gamma} + C_{pr}\mathcal{D}_{\tilde{X}_i^j}(f_i)^2. \end{aligned} \tag{2.37}$$

Here C_{pr} is the constant from the proximity condition. This estimate is valid for dense enough input, which we can always achieve by going to a finer sampling level since f is continuous. Analogously, if $i \leq j$,

$$\|(f_{i+1} - S_if_i)|_{\tilde{X}_{i+1}^j}\| \leq 2C_f\lambda^{i\gamma} + C_{pr}\mathcal{D}_{\tilde{X}_i^j}(f_i)^2. \tag{2.38}$$

From (2.37) and (2.38) we conclude (2.36) if we know the estimates

$$\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq C2^{-(i-j)\gamma/2}\lambda^{j\gamma/2} \qquad (i > j), \tag{2.39}$$

$$\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq C\lambda^{i\gamma/2} \qquad (i \leq j), \tag{2.40}$$

for some constant $C > 0$. We are thus left with proving (2.39) and (2.40). We write, for $i > j > i_0$,

$$\begin{aligned} f_i =&(f_i - S_{i-1}f_{i-1}) + \ldots + (S_{i-1,j+1}f_{j+1} - S_{i-1,j}f_j) + (S_{i-1,j}f_j - S_{i-1,j-1}f_{j-1}) \\ &+ \ldots + (S_{i-1,i_0+1}f_{i_0+1} - S_{i-1,i_0}f_0) + S_{i-1,i_0}f_{i_0}. \end{aligned} \tag{2.41}$$

By Lemma 2.11 in [25], there is a constant C_S such that for any subdivision level k and input p_k on level k,

$$\begin{aligned} &\mathcal{D}_{\tilde{X}_i^j}(S_{i-1,k}p_k) \le C_S 2^{-(i-k)}\mathcal{D}_{\tilde{X}_k^j}(p_k) && (i \ge k > j), \\ &\mathcal{D}_{\tilde{X}_i^j}(S_{i-1,k}p_k) \le C_S 2^{-(i-j)}\lambda^{j-k}\mathcal{D}_{\tilde{X}_k^j}(p_k) && (i > j > k), \\ &\mathcal{D}_{\tilde{X}_i^j}(S_{i-1,k}p_k) \le C_S \lambda^{i-k}\mathcal{D}_{\tilde{X}_k^j}(p_k) && (i \le j \le k). \end{aligned} \tag{2.42}$$

Furthermore,

$$\begin{aligned} \mathcal{D}_{\tilde{X}_k^j}(f_k - S_{k-1}f_{k-1}) &\le \mathcal{D}_{\tilde{X}_k^j}(f_k - T_{k-1}f_{k-1}) + \mathcal{D}_{\tilde{X}_k^j}(T_{k-1}f_{k-1} - S_{k-1}f_{k-1}) \\ &\le 2\|(f_k - T_{k-1}f_{k-1})|_{\tilde{X}_k^j}\| + 2\|(T_{k-1}f_{k-1} - S_{k-1}f_{k-1})|_{\tilde{X}_k^j}\| \end{aligned}$$

We use this and (2.42) to estimate the single terms in the telescoping sum (2.41). For the control sets $\tilde{X}_i^j$, where $i > j > i_0$, we get

$$\begin{aligned} \mathcal{D}_{\tilde{X}_i^j}(f_i) \le\ & 2\|(f_i - T_{i-1}f_{i-1})|_{\tilde{X}_i^j}\| + 2\|(T_{i-1}f_{i-1} - S_{i-1}f_{i-1})|_{\tilde{X}_i^j}\| \\ &+ 2\sum\nolimits_{k=j+1}^{i-1} C_S 2^{-(i-k)}(\|(f_k - T_{k-1}f_{k-1})|_{\tilde{X}_k^j}\| + \|(T_{k-1}f_{k-1} - S_{k-1}f_{k-1})|_{\tilde{X}_k^j}\|) \\ &+ 2\sum\nolimits_{k=i_0+1}^{j} C_S 2^{-(i-j)}\lambda^{j-k}(\|(f_k - T_{k-1}f_{k-1})|_{\tilde{X}_k^j}\| + \|(T_{k-1}f_{k-1} - S_{k-1}f_{k-1})|_{\tilde{X}_k^j}\|) \\ &+ C_S 2^{-(i-j)}\lambda^{j-i_0}\mathcal{D}_{\tilde{X}_{i_0}^j}(f_{i_0}). \end{aligned}$$

Using (2.35), proximity and again (2.42), we further obtain

$$\begin{aligned} \mathcal{D}_{\tilde{X}_i^j}(f_i) \le\ & 2C_f 2^{-(i-j)\gamma}\lambda^{j\gamma} + 2C_S C_f\Big(\sum\nolimits_{k=j+1}^{i-1} 2^{-(i-k)}2^{-(k-j)\gamma}\lambda^{j\gamma} + \sum\nolimits_{k=i_0+1}^{j} 2^{-(i-j)}\lambda^{j-k}\lambda^{k\gamma}\Big) \\ &+ 2C_{pr}\mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1})^2 + 2C_S C_{pr}\Big(\sum\nolimits_{k=j+1}^{i-1} 2^{-(i-k)}\mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2 \\ &\qquad + \sum\nolimits_{k=i_0+1}^{j} 2^{-(i-j)}\lambda^{j-k}\mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2\Big) \\ &+ C_S 2^{-(i-j)}\lambda^{j-i_0}\mathcal{D}_{\tilde{X}_{i_0}^j}(f_{i_0}) =: A + B + C. \end{aligned} \tag{2.43}$$

Here the symbols A, B, C refer to the first line, second plus third line, and forth line in (2.43). Analogously, we obtain, for $j \le i_0$,

$$\begin{aligned} \mathcal{D}_{\tilde{X}_i^j}(f_i) \le\ & 2C_f 2^{-(i-j)\gamma}\lambda^{j\gamma} + 2C_S C_f \sum\nolimits_{k=i_0+1}^{i-1} 2^{-(i-k)}2^{-(k-j)\gamma}\lambda^{j\gamma} \\ &+ 2C_{pr}\mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1})^2 + 2C_S C_{pr}\sum\nolimits_{k=i_0+1}^{i-1} 2^{-(i-k)}\mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2 \\ &+ C_S 2^{-(i-i_0)}\mathcal{D}_{\tilde{X}_{i_0}^j}(f_{i_0}) =: A + B + C. \end{aligned} \tag{2.44}$$

Furthermore, for $j \geq i$,

$$\begin{aligned}\mathcal{D}_{\tilde{X}_i^j}(f_i) &\leq 2C_f\lambda^{i\gamma} + 2C_SC_f\sum\nolimits_{k=i_0+1}^{i}\lambda^{i-k}\lambda^{k\gamma} \\ &\quad + 2C_{pr}\mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1})^2 + 2C_SC_{pr}\sum\nolimits_{k=i_0+1}^{i}\lambda^{i-k}\mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2 \\ &\quad + C_S\lambda^{i-i_0}\mathcal{D}_{\tilde{X}_{i_0}^j}(f_{i_0}) =: A + B + C. \end{aligned} \tag{2.45}$$

We estimate the terms called 'A' in the formulas (2.43), (2.44), and (2.45). Since $\gamma/2 < 1$, we can extract the terms $2^{-(i-j)\gamma/2}\lambda^{j\gamma/2}$ in (2.43) and (2.44) and the term $\lambda^{i\gamma/2}$ in (2.45), respectively, from the sum. The remaining summands, in case (2.43), read

$$\sum\nolimits_{k=j+1}^{i-1} 2^{-(i-k)(1-\gamma/2)}2^{-(k-j)\gamma/2}\lambda^{j\gamma/2} + \sum\nolimits_{k=i_0+1}^{j} 2^{-(i-j)(1-\gamma/2)}\lambda^{(j-k)(1-\gamma/2)}\lambda^{k\gamma/2}.$$

This is bounded from above by the corresponding geometric series (We get rid of the dependence on the index j by estimating $1/2$ and λ from above by $\max(2^{-1}, \lambda) < 1$.). The series yields an upper bound independent of i, j, and i_0. Proceeding in an analogous way for (2.44) and (2.45) yields a constant $D > 0$, independent of i, j, and i_0 such that

$$A \leq D2^{-(i-j)\gamma/2}\lambda^{j\gamma/2} \qquad \text{in case of (2.43) and (2.44)}, \tag{2.46}$$

$$A \leq D\lambda^{i\gamma/2} \qquad \text{in case of (2.45)}. \tag{2.47}$$

We are ready to estimate $\mathcal{D}_{\tilde{X}_i^j}(f_i)$. We choose i' such that

$$18D^2C_SC_{pr}\max(\lambda, 2^{-1})^{(i'-2)\gamma/2}(1 - \max(\lambda, 2^{-1})^{\min(1-\gamma/2,\gamma/2)})^{-1} < 1. \tag{2.48}$$

This reason for this choice becomes clear later on.

We notice that f is continuous, thus uniformly continuous because of its compact support. Therefore, we can choose the initial level i_0 for our estimates such that

$$\mathcal{D}(f_i) \leq \min(1, (18D^2C_SC_{pr})^{-1}(i'+1)^{-1}\min(\lambda, 2^{-1})^{i'}) =: D' \quad \text{for all } i \geq i_0. \tag{2.49}$$

We show that, for all $i \geq i_0$,

$$\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq \min(3D2^{-(i-j)\gamma/2}\lambda^{(j-i_0)\gamma/2}, D') \qquad (i > j \geq i_0), \tag{2.50}$$

$$\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq \min(3D2^{-(i-i_0)\gamma/2}, D') \qquad (j < i_0), \tag{2.51}$$

$$\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq \min(3D\lambda^{-(i-i_0)\gamma/2}, D') \qquad (j \geq i). \tag{2.52}$$

Once (2.50) – (2.52) is proved, we obtain (2.36) and the corresponding statement for $i < j$ by enlarging constants (i_0 is a fixed integer, so we can multiply with const $(\cdot)^{i_0\gamma/2}$).

It remains to show (2.50) – (2.52) for which we use induction on i. The case $i = i_0$, is clear. We assume that (2.50) – (2.52) hold for the values $i_0, \ldots, i-1$. Using the decompositions (2.43), (2.44), and (2.45) we get, for $i \geq i_0 > j$,

$$\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq A + B + C \leq D2^{-(i-j)\gamma/2}\lambda^{-(j-i_0)\gamma/2} + B + D2^{-(i-j)\gamma/2}\lambda^{-(j-i_0)\gamma/2}. \tag{2.53}$$

For $i > j \geq i_0$, $\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq 2D2^{-(i-i_0)\gamma/2} + B$, and for $j \geq i$, $\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq 2D\lambda^{-(i-i_0)\gamma/2} + B$.

We only consider the case $i \geq i_0 > j$, since the other cases are analogous. In this case, it remains to show that $B \leq D2^{-(i-j)\gamma/2}\lambda^{-(j-i_0)\gamma/2}$. We use the induction hypotheses and see

that for $i_0 + i' < j$,

$$\begin{aligned} B &\le 2C_S C_{pr}\Big(\sum\nolimits_{k=j+1}^{i} 2^{-(i-k)} \mathcal{D}_{\tilde{X}^j_{k-1}}(f_{k-1})^2 + \sum\nolimits_{k=i_0+i'+1}^{j} 2^{-(i-j)}\lambda^{j-k} \mathcal{D}_{\tilde{X}^j_{k-1}}(f_{k-1})^2 \\ &\qquad + \sum\nolimits_{k=i_0+1}^{i_0+i'} 2^{-(i-j)}\lambda^{j-k} \mathcal{D}_{\tilde{X}^j_{k-1}}(f_{k-1})^2 \Big) \\ &\le 18D^2 C_S C_{pr}\Big(\sum\nolimits_{k=j+1}^{i} 2^{-(i-k)} 2^{-(k-1-j)\gamma} \lambda^{(j-i_0)\gamma} + \sum\nolimits_{k=i_0+i'+1}^{j} 2^{-(i-j)}\lambda^{j-k}\lambda^{(k-1-i_0)\gamma} \Big) \\ &\qquad + 2C_S C_{pr}(i'+1) 2^{-(i-j)}\lambda^{j-(i_0+i')} D'. \end{aligned}$$

We extract factors in the sums and apply the definition of D' :

$$\begin{aligned} B &\le (C_S C_{pr}) 18D^2 (2^{-(i-j)\gamma/2}\lambda^{(j-i_0)\gamma/2} \max(\lambda, 1/2)^{-\gamma/2} \\ &\qquad \Big(\sum_{k=j+1}^{i} 2^{-(i-k)(1-\frac{\gamma}{2})} 2^{-(k-1-j)\frac{\gamma}{2}} \lambda^{(j-i_0)\frac{\gamma}{2}} + \sum_{k=i_0+i'+1}^{j} 2^{-(i-j)(1-\frac{\gamma}{2})} \lambda^{(j-k)(1-\frac{\gamma}{2})} \lambda^{(k-1-i_0)\frac{\gamma}{2}} \Big) \\ &\quad + 2C_S C_{pr} (18D^2 C_S C_{pr})^{-1} (i'+1)(i'+1)^{-1} 2^{-(i-j)} \lambda^{j-(i_0+i')} \min(\lambda, 2^{-1})^{i'}. \end{aligned}$$

We further estimate

$$\begin{aligned} B &\le (C_S C_{pr})(18D^2 \max(\lambda, 1/2)^{-\frac{\gamma}{2}} i'^{\frac{\gamma}{2}}) 2^{-(i-j)\frac{\gamma}{2}} \lambda^{(j-i_0)\frac{\gamma}{2}} \Big(\sum_{k=0}^{\infty} \max(\lambda, 2^{-1})^{\min(1-\frac{\gamma}{2}, \frac{\gamma}{2})k} \Big) \\ &\qquad + \tfrac{1}{18} 2^{-(i-j)} \lambda^{j-i_0} \\ &\le C_S C_{pr} 2^{-(i-j)\gamma/2} \lambda^{(j-i_0)\frac{\gamma}{2}} + \tfrac{1}{18} 2^{-(i-j)} \lambda^{j-i_0} \le D 2^{-(i-j)\frac{\gamma}{2}} \lambda^{(j-i_0)\frac{\gamma}{2}} \end{aligned} \tag{2.54}$$

For $i_0 + i' \ge j$ as well as the other two cases (2.50) and (2.52) one proceeds in an analogous way. This completes the proof of Theorem 2.3.

□

We conclude with the proofs of Corollary 2.4 and Corollary 2.5.

Proof of Corollary 2.4. By Theorem 2.3 a local proximity condition (1.10) must be shown for a geometric (bundle) analogue of S given by (1.6). In the bundle framework, such a condition is shown in [7], for particular instances, including the examples (1.5) and (1.2), see also [23, 6, 25]. Note that a 'global' proximity condition is stated in [7], but actually our local version (1.10) is proved. □

Proof of Corollary 2.5. The smoothness index $\omega = \min(\nu, \nu')$ was defined by (2.4). By [30], $\nu' = 2$, and since $\nu < 2$, $\omega = \nu$. Also, by [30] the subdominant eigenvalue λ of the subdivision matrix equals $1/2$. So plugging $\lambda = 1/2$ into (2.3) completes the proof. □

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