Nonlinear subdivision schemes on irregular meshes

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Abstract

The present article deals with convergence and smoothness analysis of geometric, nonlinear, subdivision schemes in the presence of extraordinary points. We discuss when the existence of a proximity condition between a linear scheme and its nonlinear analogue implies convergence of the nonlinear scheme (for dense enough input data). Furthermore, we obtain C^1 smoothness of the nonlinear limit function in the vicinity of an extraordinary point over Reif's characteristic parametrisation. The results apply to the geometric analogues of well known subdivision schemes like Doo-Sabin or Catmull-Clark schemes.

keywords: nonlinear subdivision, extraordinary point, irregular mesh.

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1 Introduction

Linear subdivision schemes on polygonal meshes are frequently used in Computer Graphics and Geometric Modeling. An overview of the use of subdivision in these areas is given in [19]. Subdivision schemes also have other applications; for instance, linear subdivision schemes defined on regular meshes with values in $\mathbb{R}^d$ are applied to produce scaling functions in wavelet analysis.

The first linear schemes for not necessarily regular meshes go back to Catmull and Clark [1] and Doo and Sabin [3], whose schemes are the first examples of primal resp. dual quadrilateral based schemes.

The analysis of linear stationary subdivision schemes in the neighborhood of extraordinary points is covered in depth in the book [10]; We would also like to mention the paper [18]. This extension of subdivision to non-regular meshes is important, since many closed surfaces in $\mathbb{R}^3$ do not admit a covering by a regular mesh.

A framework for the analysis of geometric, nonlinear, subdivision schemes by means of so-called proximity inequalities has been introduced by Wallner and Dyn in [15] for the univariate case, and in the multivariate regular grid setting by Grohs [4]. For an overview on previous ways of analysing nonlinear subdivision schemes we refer to [15]. Such schemes are designed to deal with data that live in a nonlinear geometry such as a Lie group or a Riemannian manifold. Examples are samples of poses of a rigid body in

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motion, or diffusion tensor images (where the data are positive definite matrices). Other instances of geometric data in connection with subdivision are given in [14].

In this article we deal with an essential part of the theory which was missing until now: Convergence and C^1 smoothness of nonlinear subdivision rules for irregular meshes. We show that a certain class of such schemes converge and produce C^1 limit functions in the vicinity of an extraordinary point w.r.t. Reif's characteristic parametrisation. This analysis is based on a local proximity inequality similar to that in [15]: If a nonlinear scheme is in proximity with a linear scheme which converges resp. produces C^1 limit functions, then the nonlinear scheme does the same for sufficiently dense input data.

We begin our article by gathering facts on linear subdivision schemes and building up a framework for the following convergence and smoothness analysis for nonlinear schemes. We apply our results to geometric schemes. This paper also introduces a new geometric analogue which is well suited to subdivision on meshes. We conclude with an application, namely the modelling of C^1 functions between manifolds by means of control points.

1.1 Linear Subdivision Schemes and Setup

Let us establish the vocabulary we need later on. We review linear subdivision and introduce a setup in the spirit of Reif's [13] framework near extraordinary points. We have to incorporate some discrete component since, in the nonlinear case, we do not have a finite set of a priori known surface patches.

We distinguish between a combinatorial *mesh connectivity* $(\mathcal{V}, \mathcal{E}, \mathcal{F})$, and a *mesh* which consists of a mesh connectivity and a mapping h, defined in the vertex set $\mathcal{V}$, so that $h(v)$ represents the geometric position of a vertex. In a mesh connectivity we use the notation $\mathcal{N}_n(v)$ and $\mathcal{N}_n(F)$ for the n-ring of a vertex v, and of a face F, respectively.

A subdivision scheme S consists of a topological and a geometric refinement rule. The topological rule generates, for a given connectivity $(\mathcal{V}_0, \mathcal{E}_0, \mathcal{F}_0)$, a new connectivity $(\mathcal{V}_1, \mathcal{E}_1, \mathcal{F}_1)$. We consider the two well known categories of *primal* and *dual* topological rules, of which the schemes of Catmull/Clark and Doo/Sabin are examples. The geometric rule computes new vertex positions from old ones. In the case of linear subdivision, we consider *affine invariant* rules, meaning that a new vertex position $h_1(w)$ is an affine combination of finitely many previous ones:

$$h_1(w) = \sum_{v \in \mathcal{V}_0} \alpha_{v,w} h_0(v). \tag{1.1}$$

We call the mapping $v \mapsto \alpha_{v,w}$ the stencil of w, and denote its *support* by $\mathrm{supp}_S(w)$. We further require that the rule only depends on the mesh connectivity in a local neighbourhood of globally fixed size (see [18] for details).

Both primal and dual rules have the property that after a few iterations, the greater part of the connectivity becomes regular, i.e., faces and vertices have valence 4. However, there are remaining isolated singularities,*extraordinary vertices* in the primal case, and *extraordinary faces* in the dual case, which are surrounded by regular connectivity. It is well known that by the locality of the geometric rule, it is, in the linear case, enough to consider a mesh with only one irregular vertex/face.

For the purpose of analysis of subdivision at these singular locations, fix an integer $k > 2$ which stands for the valence of the extraordinary vertex or face. We glue k copies

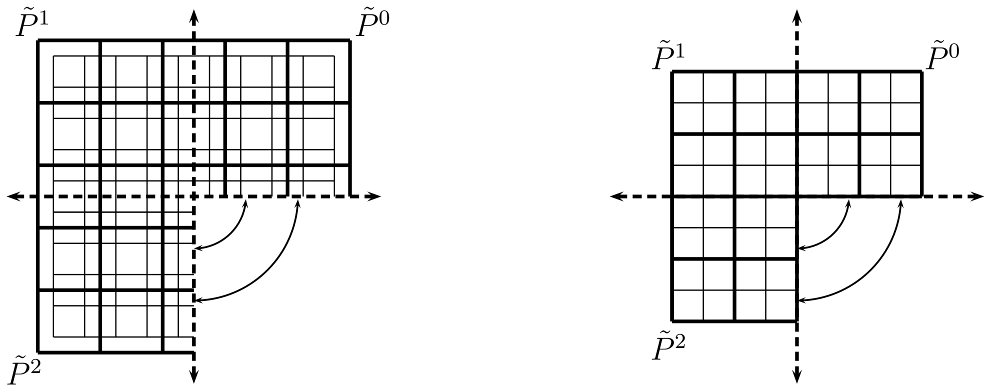

Figure 1: A cut-out of the domain $\tilde{P}$ and the 0th and 1st 3-regular connectivities in the dual case (left) and primal case (right)

$\tilde{P}^0, \dots, \tilde{P}^{k-1}$ of the positive quadrant together by identifying the y-axis of $\tilde{P}^j$ with the x-axis of $\tilde{P}^{j+1}$ (indices modulo k). The result is denoted by $\tilde{P} = [0, \infty[\times [0, \infty[\times \mathbb{Z}/k\mathbb{Z}$. We refer to $\tilde{P}^j$ as its j-th sector. The segment $P_i^j(r)$, depending on the 'radius' r is the set $\{(x, y, j) \in \tilde{P} : \ 2^{-i-1}r \leq \max(x, y) \leq 2^{-i}r\}$, and the ring $P_i(r)$ is the union of all segments $P_i^j(r)$. $\tilde{P}$ becomes a metric space by defining the distance of points by the length of the shortest path which connects them, with the metric in the single sectors being that of $\mathbb{R}^2$. The mapping R^j is given by keeping the j-th quadrant and rotating the $(j+1)$-st by 90 degrees. Thus, it bijectively maps two successive sectors to the upper half plane.

The discrete component, the k-regular connectivity $(M_n^k, \mathcal{E}_n, \mathcal{F}_n)$ at subdivision level n is defined as follows. In the primal case, we define $M_n^k \subset \tilde{P}$ by $M_n^k = 2^{-n}(\mathbb{N}_0 \times \mathbb{N}_0) \times \mathbb{Z}/k\mathbb{Z}$. In the dual case, we let $M_n^k = [(2^{-n-1}, 2^{-n-1}) + 2^{-n}(\mathbb{N}_0 \times \mathbb{N}_0)] \times \mathbb{Z}/k\mathbb{Z}$. Via R^j, parts of M_n^k are mapped to vertices of the regular grid $2^{-n}\mathbb{Z}^2$ (to a translated regular grid in the dual case.) We choose $\mathcal{E}_n$ and $\mathcal{F}_n$, such that it agrees with part of the regular grid under R^j. So we obtain a connectivity with one single valence k vertex, resp. face in the dual case. In both cases, M_{n+1}^k arises from M_n^k by dilation with factor 2, see Figure 1. The action of a subdivision scheme S on a k-regular input mesh is interpreted in the following way: It transforms vertex data $h : M_n^k \to \mathbb{R}^d$ at level n to new vertex data $S_n h : M_{n+1}^k \to \mathbb{R}^d$. We distinguish the operations on different levels since we find it more convenient for the following analysis. For the dilation operator D defined by $Df(x) = f(2x)$ we obviously have $S_n \circ D = D \circ S_{n+1}$. We use the notation $S_{n,m} = S_n \cdots S_m$ (if $n < m$, $S_{n,m}$ is the identity).

We say that a rule S *converges* on the k-regular mesh, if for bounded initial data p_0, i.e., $p_0 \in l^\infty(M_0^k, \mathbb{R}^d)$, there is a uniformly continuous f, i.e., $f \in C_u(\tilde{P}, \mathbb{R}^d)$, such that $\|f|M_i^k - S_{i-1,0}p_0\|_\infty$ converges to 0, as $i \to \infty$. For the limit we use the notation $S_{\infty,0}p_0$. If data p_n is on level n, we write $S_{\infty,n}p_n$ for the limit.

For any compact subset A of the domain $\tilde{P}$, a limit depends only on finitely many vertex data at level j, for any given level j. We use the notation $\mathrm{ctrl}^j(A)$ for the vertices of M_j^k which influence limit functions restricted to$A$.

Away from the extraordinary vertex or face, the mesh is regular. For regular meshes, the subdivision scheme is seen as an operator on $l^\infty(\mathbb{Z}^2, \mathbb{R}^d)$, given by $Sp = \sum_{\alpha \in \mathbb{Z}^2} a(\cdot - 2\alpha)p(\alpha)$, where the *mask* a has finite support. For a detailed presentation, we refer to

[2]. The linear bounded operator $\nabla : l^\infty(\mathbb{Z}^2, \mathbb{R}^d) \to (l^\infty(\mathbb{Z}^2, \mathbb{R}^d))^2$ is given by $\nabla p(m) = (p(m+e_1) - p(m),\, p(m+e_2) - p(m))$, where e_1 and e_2 are the coordinate directions. By affine invariance, the derived scheme $S^{[1]}$, given by $S^{[1]}\nabla = 2\nabla S$, exists.

Proposition 1.1. *Linear S as above converges on the regular mesh if and only if there is $C \geq 1$ and $0 < \gamma < 1$, such that*

$$\|\nabla S^k p\|_\infty \leq C\gamma^k \|\nabla p\|_\infty, \qquad \textit{for all } p \in l^\infty(\mathbb{Z}^2, \mathbb{R}^d).$$

If the derived scheme $S^{[1]}$ converges, we can choose $\gamma = 1/2$.

For example, the derived scheme converges, if S is stable and produces C^1 limits. The first part of the above statement is well known, see e.g. [2]. We did not find a proof of the exact statement of the second part, so we briefly explain how it can be seen: For $f \in l^\infty(\mathbb{Z}^2, \mathbb{R}^d)$, $\|S^{[1]k}\nabla f\|_\infty \leq D$, with D independent of k. Restrict f to $B = \{-n, \dots, n\}^2$, where n is big enough such that B controls the limit on the unit square. We apply the Banach-Steinhaus-Theorem to the operators $(S^{[1]})^k$ restricted to the finite dimensional space of sequences vanishing outside $3B$ and on 0. This yields that $\|2^k \nabla S^k f'\|_\infty = \|S^{[1]k}\nabla f'\|_\infty \leq C'\|\nabla f'\|_\infty$, for all such sequences f'. Here C' is independent of f'. For general f, we find f', such that on B, $\nabla f = \nabla f'$.

Our Setup. We impose the following conditions on linear subdivision schemes. The major restriction in contrast to Reif's setup for standard algorithms [10] comes from the fact that we do not take the point of view of iteratively generating control points of surface patches. In the nonlinear case, this view is not possible since such a finite dimensional space of patches is not available in general. This also explains that our notion of a subdivision matrix, given below, differs from [10]. For us, a standard subdivision scheme S is a linear scheme with the following properties:

(1) For regular connectivity, the derived scheme $S^{[1]}$ converges.

(2) There is a 'radius' $r > 0$, such that the control sets $\mathrm{ctrl}^i(P_i^j(r))$ are vertices of a regular connectivity. The subdivision matrix A maps data on $\mathrm{ctrl}^i(P_i(r))$, controlling the ring P_i, to data on $\mathrm{ctrl}^{i+1}(P_{i+1}(r))$.

(3) The subdivision matrix A has the single eigenvalue 1 and algebraically and geometrically double subdominant eigenvalue $\lambda \in\,]0, 1[$. The characteristic map, defined below, is regular and injective.

We simply write P_i^j instead of $P_i^j(r)$. Examples of schemes which meet these requirements are the *generalized Lane-Riesenfeld schemes* [20], of which the classical *Doo-Sabin* [3] and *Catmull-Clark scheme* [1] are particular examples. Those two schemes are generalized and analysed in [11]. An example of an interpolatory scheme is *Kobbelt's interpolatory quad scheme* [8], which was analysed by Zorin in [17].

The notion of a *characteristic map* has been introduced by Reif in [13]. Our definition is slightly different and follows Prautzsch [12]. The limit function of subdivision on P, which is the union of all rings P_i and 0, is determined by data on $\mathrm{ctrl}^0(P_0)$. We choose two linearly independent eigenvectors to the subdominant eigenvalue of A. (Acually, all such choices of eigenvectors essentially lead to the same characteristic map.) Each one determines a component of 2D input data on $\mathrm{ctrl}^0(P_0)$. The limit function $\chi : P \to \mathbb{R}^2$ of these data is called the characteristic map.

The following theorem is due to Reif [13]. A proof which immediately generalizes to $\mathbb{R}^d$, $d \geq 2$, has been given by Prautzsch [12].

Theorem 1.2. *For a standard scheme S and input data p_0 on $\mathrm{ctrl}^0(P_0)$ with values in $\mathbb{R}^d$, let $S_{\infty,0}p_0$ be the limit function of subdivision. Then the map $S_{\infty,0}p_0 \circ \chi^{-1} : \chi(P_0) \to \mathbb{R}^d$ is C^1. For almost all input data p_0, the image $S_{\infty,0}p_0(P)$ is a two-dimensional submanifold of $\mathbb{R}^d$ locally around the (extraordinary) limit point $S_{\infty,0}p_0(0)$.*

1.2 Geometric Subdivision Schemes and Results

Geometric subdivision schemes are designed to handle data in smooth manifolds. Except for subdivision by intrinsic means as defined below, our examples were also considered by Wallner and Dyn [15], and Grohs [4]. We refer to these papers for questions of well definedness and for a survey of previous work.

We consider a linear subdivision scheme S and obtain a *geometric* scheme T analogous to S as follows: T produces the same mesh connectivity as S and the geometric rule is modified in order to deal with the geometric data. Using arbitrary *base points* $x(w)$, the linear rule (1.1) can be rewritten as

$$h_1(w) = x(w) + \sum\nolimits_v \alpha_{v,w}(h_0(v) - x(w)). \tag{1.2}$$

For the *log-exp analogue* the data is supposed to take values in a Lie group, a Riemannian manifold or a symmetric space, see [16]. The $+$ and $-$ operations in (1.2) are replaced by *exp* and its inverse, which are available in that geometries. We obtain

$$h_1(w) = \exp_{x(w)}(\sum\nolimits_v \alpha_{v,w} \ \exp^{-1}_{x(w)}(h_0(v)),$$

with base points $x(w)$ in the manifold. The choice of base points should match with the connectivity of the mesh: a vertex of a refined mesh is combinatorically associated with a vertex, edge or face of the original mesh. It makes sense to let w's ancestor determine $x(w)$, e.g. as intrisic edge midpoint or face midpoint. One possible face midpoint is the midpoint of diagonals.

The *intrinsic mean analogue* processes data in Riemannian manifolds. The idea of an intrinsic mean in a Riemannian manifold M, also called Karcher mean or Riemannian center of mass, goes back to Cartan. For details, we refer to [6]. In the context of 'meshless geometric subdivision', intrinsic midpoints of surfaces were used in [9]. We briefly explain the idea of the intrinsic mean analogue: For finitely many points $\{x_i\}_{1\leq i \leq n}$, which are contained in a small enough ball, and coefficients $\{\alpha_i\}_{1\leq i\leq n}$, summing up to one, there is a unique point x, such that

$$\sum\nolimits_{i=1}^n \alpha_i \ d(x_i, x)^2 = \min_{y\in M} \sum\nolimits_{i=1}^n \alpha_i \ d(x_i, y)^2. \tag{1.3}$$

Here d is the Riemannian metric in M. The point x is called the intrinsic mean of the points x_i w.r.t. the weights α_i. For estimates on the sizes of the above Riemannian balls we refer to Kendall [7]. We can define the intrinsic means analogue by taking the weights $\alpha_{v,w}$ of (1.2) in (1.3). It is well known that the sequence $\{y_j\}_{j\in\mathbb{N}_0}$, defined by $y_{j+1} = \exp_{y_j}(\sum_{i=1}^n \alpha_i \ \exp_{y_j}^{-1}(x_i))$ converges to the intrinsic mean, if we choose the start point y_0 in the small ball. Thus, on the one hand, the action of the log-exp analogue at a point in a Riemannian manifold can be interpreted as first step in the iteration to the intrinsic mean. On the other hand, the intrinsic mean analogue can be interpreted as log-exp analogue with a very special choice of base points, namely the means itself, since $\exp_x(\sum_{i=1}^n \alpha_i \ \exp_x^{-1}(x_i)) = x$. A major advantage of using intrinsic means is that the symmetries of the respective linear scheme are preserved.

The *geodesic analogue* and the *projection analogue* are discussed in [15].

A framework for the analysis of convergence and C^1 smoothness of geometric schemes was built by Wallner and Dyn [15] in the case of curve subdivision and by Grohs [4] for regular grids. Roughly speaking, convergence and smoothness issues for these schemes are dealt with by locally embedding the manifold into $\mathbb{R}^d$. Under this embedding the according scheme T for manifold subdivision is shown to meet a proximity condition with a linear scheme S, which is slightly weaker than the following one.

Definition 1.3. *Let $M \subset \mathbb{R}^d$, $\delta > 0$. We consider (not necessarily linear) subdivision schemes S and T with the same topological rule. Let h_0 be the positioning function of the input mesh with values in M, such that the distance of neighbouring vertices is smaller than δ. Assume that both schemes are defined for such h_0. Then S satisfies a local (M, δ) proximity condition if there is a constant C, such that*

$$\|h_1^S(w) - h_1^T(w))\| \le C \sup_{v_1,v_2 \in \operatorname{supp}_S(w)} \|h_0(v_1) - h_0(v_2)\|^2. \tag{1.4}$$

Here h_1^S and h_1^T are the results of refinement using S and T, respectively.

The main result of this paper is the following. It is proved at the very end of Section 2.

Theorem 1.4. *For dense enough input, the nonlinear analogues mentioned above converge for any mesh with an upper bound on the valence of vertices and faces. They produce C^1 limits near extraordinary points w.r.t. the characteristic parametrisation.*

2 Convergence and Smoothness Analysis

Our analysis considers a nonlinear scheme T, which is related to a linear scheme S as in Definition 1.3. By the locality of the proximity condition (1.4), T is also a local scheme, and a new vertex generated by T depends only on the old ones in the support of the according stencil of S.

To avoid additional notation, we want to restrict our analysis of a nonlinear scheme T to k-regular meshes. In contrast to the linear case, this does not immediately work. A minor problem lies in the 'dense enough' assuption for input data, which can be overcome as follows: If we have an input mesh with an upper bound on the valence of faces and vertices, we can postulate the input data even denser, such that after the first subdivision steps, the mesh is still dense enough near the extraordinary object, but the connectivity around the latter is k-regular. Therefore, a convergence statement for k-regular meshes implies a convergence statement for the general case.

The main statements of this section are Theorem 2.4 and Theorem 2.10. Theorem 1.4 is proved at the very end of this section.

For interpreting a k-regular mesh as a function we have defined the discrete domains M_n^k, $n \in \mathbb{N}_0$. We introduce the following (nonlinear) difference operator:

Definition 2.1. *Let $p_n \in l^\infty(M_n^k, \mathbb{R}^d)$. For $B \subset M_n^k$, we define $\Delta_B p_n(v) = \sup\{\|p_n(v) - p_n(w)\|_{\mathbb{R}^d} : w \in \mathcal{N}_1(v) \cap B\}$. We let*

$$\mathcal{D}_B(p_n) := \sup\{\Delta_B p_n(v) : \; v \in B\},$$

and we drop the index B, if $B = M_n^k$.

The quantity $\mathcal{D}_B$ obviously satisfies the triangle inequality. Now, we let the class $P_{M,\delta}$ be all functions p_n, defined on some M_n^k ($n \in \mathbb{N}_0$), with values in $M \subset \mathbb{R}^d$, and $\mathcal{D}(p_n) \leq \delta$ ($\delta > 0$; M can e.g. be chosen as a submanifold or an open set). Then the local proximity condition reads: There is $C > 0$ such that for all $n \in \mathbb{N}_0$, and all $p_n \in P_{M,\delta}$,

$$\|S_n p_n(v) - T_n p_n(v))\| \leq C \sup\nolimits_{v_1, v_2 \in \mathrm{supp}_S(v)} \|p_n(v_1) - p_n(v_2)\|^2.$$

Note that control sets w.r.t. S are also contol sets for T. We consider the sequence of sets $V_n = M_n^k$, $V_n = \mathrm{ctrl}^n(P_i^j)$ or $V_n = \mathrm{ctrl}^n(P_n)$, where $n = 0, 1, 2, \ldots$. For those sets, a local proximity condition implies that there is a constant F with

$$\|S_n p_n(v) - T_n p_n(v))\|_{\infty, V_{n+1}} \leq F(\mathcal{D}_{V_n}(p_n))^2, \tag{2.1}$$

for $p_n \in P_{M,\delta}$. This follows immediately from the locality of S, using the triangle inequality and the fact that $(a+b)^2 \leq 2\,(a^2 + b^2)$ for $a, b \in \mathbb{R}$.

We state a technical lemma which is a key ingredient in the proof of both the convergence and smoothness result. The sequence g_n in the lemma should be thought of as the data the nonlinear scheme produces.

Lemma 2.2. *Let S be a standard scheme. Let $V_n \subset M_n^k$ ($n = 0, 1, 2, \ldots$) be a sequence of subsets, such that subdivision of data p_n on V_n determines $S_n p_n$ on V_{n+1}. We assume that there are $C \geq 1$ and $\gamma \in (0,1)$ such that for all $n \in \mathbb{N}_0$ and $p_n \in l^\infty(V_n, \mathbb{R}^d)$, and all $k \geq n$,*

$$\mathcal{D}_{V_k}(S_{k-1,n} p_n) \leq C \gamma^{k-n} \mathcal{D}_{V_n}(p_n).$$

Let $m \in \mathbb{N}$ and suppose there is $C' > 0$ such that for a sequence $\{g_n\}_{n=0}^{m+1}$ with $g_n \in l^\infty(V_n, \mathbb{R}^d)$ we have the inequalities:

$$\|g_{n+1} - S_n g_n\|_\infty \leq C' \gamma (\mathcal{D}_{V_n}(g_n))^2 \tag{2.2}$$

for all $0 \leq n \leq m$, and

$$\mathcal{D}_{V_0}(g_0) \leq (1-\gamma)/8C'C^2. \tag{2.3}$$

Then, for all $1 \leq k \leq m$,

$$\mathcal{D}_{V_k}(g_k) \leq 2C\gamma^k \mathcal{D}_{V_0}(g_0).$$

Proof. We use induction on k. For $k = 1$ we have

$$\begin{aligned}\mathcal{D}_{V_1}(g_1) \leq & \mathcal{D}_{V_1}(g_1 - S_0 g_0) + \mathcal{D}_{V_1}(S_0 g_0) \leq 2C'\gamma(\mathcal{D}_{V_0}(g_0))^2 + C\gamma\mathcal{D}_{V_0}(g_0) \\ \leq & C(2C'\mathcal{D}_{V_0}(g_0) + 1)\gamma\mathcal{D}_{V_0}(g_0) \leq 2C\gamma\mathcal{D}_{V_0}(g_0).\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{D}_{V_k}(g_k) &\leq \sum\nolimits_{l=1}^{k} \mathcal{D}_{V_k}(S_{k-1,l} g_l - S_{k-1,l-1} g_{l-1}) + \mathcal{D}_{V_k}(S_{k-1,0} g_0) \\ &\leq \sum\nolimits_{l=1}^{k} C\gamma^{k-l}\mathcal{D}_{V_l}(g_l - S_{l-1,l-1} g_{l-1}) + \mathcal{D}_{V_k}(S_{k-1,0} g_0) \\ &\leq \sum\nolimits_{l=1}^{k} 2C\gamma^{k-l} \cdot \gamma C'(\mathcal{D}_{V_{l-1}}(g_{l-1}))^2 + C\gamma^k\mathcal{D}_{V_0}(g_0)\end{aligned}$$

We use the induction hypothesis and obtain

$$\begin{aligned}\mathcal{D}_{V_k}(g_k) \leq & \sum\nolimits_{l=1}^{k} 8CC'\gamma^{k-l+1}C^2\gamma^{2(l-1)}(\mathcal{D}_{V_0}(g_0))^2 + C\gamma^k\mathcal{D}_{V_0}(g_0) \\ \leq & C\mathcal{D}_{V_0}(g_0)\left[C^2\sum\nolimits_{l=1}^{k} 8C'\gamma^{k+l-1}\mathcal{D}_{V_0}(g_0) + \gamma^k\right] \\ \leq & C\gamma^k\mathcal{D}_{V_0}(g_0)\left[\frac{8C'C^2}{1-\gamma}\mathcal{D}_{V_0}(g_0) + 1\right] \leq 2C\gamma^k\mathcal{D}_{V_0}(g_0).\end{aligned}$$

This completes the proof. □

The next lemma expresses condition (3) on the eigenvalues of the subdivision matrix in terms of differences. It is probably not new, but we did not find it in the literature.

Lemma 2.3. *Let $A: \mathbb{R}^m \to \mathbb{R}^m$ be a matrix with single eigenvalue 1 for the eigenvector $v_1 = (1, \dots, 1)^T$, and assume that all other eigenvalues have smaller modulus. We set $\Delta'(b) := \sup_{1\leq k,j\leq m} |b_k - b_j|$ for $b \in \mathbb{R}^m$. Then for every $\varepsilon > 0$ there is $C > 1$ such that, for all $l \in \mathbb{N}$, and all $b \in \mathbb{R}^m$,*

$$\Delta'(A^l b) \leq C(|\lambda_2| + \varepsilon)^l \Delta'(b),$$

where λ_2 is a subdominant eigenvalue of A. If all eigenvalues μ with $|\mu| = |\lambda_2|$ have equal algebraic and geometric multiplicity, then we can choose $\varepsilon = 0$.

Proof. With the Jordan normal form J of A we have $AV = VJ$, where the generalized eigenvectors of A are stored in $V = (v_1, \dots, v_m)$. We assume that J is ordered by modulus and denote the dual basis of V by $\{v_i^*\}_{i=1}^m$. Then we can write

$$A^l b = A^l V V^{-1} b = \sum_{i=1}^{m} A^l v_i v_i^*(b).$$

Since $A^l v_1 = (1, \dots, 1)^T$,

$$\begin{aligned}|(A^l b)_j - (A^l b)_k| &= \left|\sum\nolimits_{i=2}^{m}[(A^l v_i)_j - (A^l v_i)_k] v_i^*(b)\right| \\ &\leq \sum\nolimits_{i=2}^{m} |(A^l v_i)_j - (A^l v_i)_k| \sup_{2\leq i\leq m} |v_i^*(b)|. \qquad (2.4)\end{aligned}$$

For estimating the first factor, consider a Jordan block D of A of size α with eigenvalue μ, eigenvector w_0, and ordered generalized eigenvectors $w_1, \dots, w_{\alpha-1}$. Then for integers $\beta > \alpha > \gamma \geq 0$ and $1 \leq j, k \leq m$,

$$|(A^\beta w_\gamma)_j - (A^\beta w_\gamma)_k| \leq \left|\mu^{\beta-\gamma} \sum_{\delta=0}^{\gamma} \tbinom{\beta}{\gamma-\delta} \mu^\delta w_\delta\right| \ \sup_{0\leq\delta\leq\gamma} |(w_\delta)_j - (w_\delta)_k|.$$

Therefore, for $\varepsilon > 0$ there is a constant $C_\mu > 0$ such that for all $\beta > \alpha > \gamma \geq 0$:

$$\left|\mu^{\beta-\gamma} \sum_{\delta=0}^{\gamma} \tbinom{\beta}{\gamma-\delta} \mu^\delta w_\delta\right| \leq C_\mu(|\mu| + \varepsilon)^\beta,$$

since the sum on the left-hand side is a polynomial in β. Thus, for $\varepsilon > 0$ there is $C_0 > 0$, such that

$$\sum_{i=2}^{m} |(A^l v_i)_j - (A^l v_i)_k| \leq C_0(|\lambda_2| + \varepsilon) \sup_{1\leq i\leq m} |(v_i)_j - (v_i)_k|.$$

For the second factor on the right hand side of (2.4) we have, for $2 \leq i \leq m$,

$$\begin{aligned} |v_i^*(b)| = |v_i^*(b - b_1 v_1)| &\leq \|v_i^*\| \|b - b_1 v_1\|_\infty \\ &\leq (\sup_{2\leq i\leq m} \|v_i^*\|) \cdot \sup_{2\leq i\leq m} |b_i - b_1|, \end{aligned}$$

where $\|\cdot\|$ is the norm of the linear functionals $v_i \in l^\infty(\{1, \dots, m\}, \mathbb{C})$. From this, the lemma follows. □

2.1 Convergence Analysis

The formulation of the following convergence theorem is rather technical, which is mainly due to the fact that definedness of T has to be guaranteed in any subdivision step.

Theorem 2.4. *Let S and T fulfill a local proximity condition w.r.t. some $P_{M,\delta}$. Assume that T_n maps $P_{M,\delta}$ to $l^\infty(M_{n+1}^k, M')$ for some M' with $M \subset M' \subset \mathbb{R}^d$. Assume further that there is $M'' \subset M$ and $\delta' > 0$ such that the δ'-neighbourhood $U_{\delta'}(M'')$ obeys $U_{\delta'}(M'') \cap M' \subset M$. Then there is $\delta'' > 0$ such that T converges for input $p_0 \in P_{M'',\delta''}$, and $S_{\infty,i+1}T_{i,0}p_0$ converges to the nonlinear limit $T_{\infty,0}p_0$ in $C_u(\tilde{P}, \mathbb{R}^d)$.*

The proof of this statement consists of two parts: The first one is to show that the contractivity of the differences of data generated by S implies the convergence of data generated by T, if the input is dense enough. The second part is to show this contractivity for a standard scheme S.

In order to show the first part, we consider the map $E : \tilde{P} \to \mathbb{R}^2$. E bijectively maps the entire $\tilde{P}$ to the plane by first squeezing the j-th quadrant into a sector of opening angle $2\pi/k$ with a shear transformation and then rotating it by an angle of $2\pi j/k$. We connect the points $E(M_n^k)$, by straight lines according to the k-regular connectivity and obtain a set of faces $\mathcal{F}_n$. For every $n \in \mathbb{N}_0$, we define the *interpolation operator* $I_n : \ l^\infty(E(M_n^k), \mathbb{R}^d) \to C_u(\mathbb{R}^2, \mathbb{R}^d)$, as follows: We split each face $F \in \mathcal{F}_n$ into triangles, each of them determined by F's barycenter and an edge. We get data for the barycenter by the barycenter of the data on the neighbouring vertices. Then we use linear interpolation on the triangles. For x, y in a face, we obviously have

$$\sup_{x,y\in F_n} \|I_n p_n(x) - I_n p_n(y)\|_{\mathbb{R}^d} \leq \mathcal{D}(p_n).$$

Note that the notion of convergence of a scheme for input p_0 is invariant under reparametrisation with the help of E. So let us consider the whole subdivision process w.r.t. $E(M_n^k) \subset \mathbb{R}^2$ instead of $M_n^k \subset \tilde{P}$.

Lemma 2.5. *Let S be a standard scheme acting as operators $S_n : l^\infty(E(M_n^k), \mathbb{R}^d) \to l^\infty(E(M_{n+1}^k), \mathbb{R}^d)$. Then $\|S_n\|$ is uniformly bounded, and each face of $\mathcal{F}_n$ is convex. Furthermore there are constants $C_1, C_2, R > 0$ such that for all $n \in \mathbb{N}_0$,*

(i) the infimum d' of distances of neighbouring vertices in $E(M_n^k)$ satisfies $C_1 2^{-n} \leq d' \leq \max_{F \in \mathcal{F}_n} \operatorname{diam} F \leq C_2 2^{-n}$;

(ii) the value $S_n p_n(v)$ is an affine combination of the local values $\{f_n(w) : w \in B(v, 2^{-n}R) \cap E(M_n^k)\}$, where $B(x,r)$ is the open ball with radius r around x.

This statement is clear by the definition of M_n^k, $\mathcal{F}_n$, and by the locality of S.

Proposition 2.6. *Let S be a standard scheme. Suppose there is $\gamma \in (0,1)$ and $C \geq 1$ such that for any $l \in \mathbb{N}$, $p_l \in l^\infty(E(M_l^k), \mathbb{R}^d)$, and $n \geq l$,*

$$\mathcal{D}(S_{n-1,l} p_l) \leq C \gamma^{n-l} \mathcal{D}(p_l). \tag{2.5}$$

Then, the sequence $\{I_n S_{n-1,l} p_l\}_{n \in \mathbb{N}_0}$ converges to $S_{\infty,l} p_l$ in $C_u(\mathbb{R}^2, \mathbb{R}^d)$. In particular, there are constants $C_B, C_I > 0$, independent of $l \in \mathbb{N}_0$ and p_l, such that

$$\begin{aligned} \|I_{l+1} S_l p_l - I_l p_l\| &\leq C_B \mathcal{D}(p_l), \\ \|S_{\infty,l} p_l|_{E(M_l^k)} - p_l\| \leq \|S_{\infty,l} p_l - I_l p_l\| &\leq C_I \mathcal{D}(p_l). \end{aligned}$$

Proof. We start by estimating $\|I_{m+1} S_m g_m - I_m g_m\|$ for general bounded g_m, defined on $E(M_m^k)$. Let $x \in \mathbb{R}^2$, and choose faces F_m of $\mathcal{F}_m$ and F_{m+1}, of $\mathcal{F}_{m+1}$, resp., which contain x. In addition denote by v_m, resp., v_{m+1}, a vertex of F_m, resp., F_{m+1}, nearest to x. Then,

$$\begin{aligned} \|I_{m+1} S_m g_m(x) - I_m g_m(x)\| &\leq \|I_{m+1} S_m g_m(x) - I_{m+1} S_m g_m(v_{m+1})\| + \\ &+ \|S_m g_m(v_{m+1}) - g_m(v_m)\| + \|I_m g_m(v_m) - I_m g_m(x)\| \\ &\leq \mathcal{D}(S_m g_m) + \mathcal{D}(g_m) + \|S_m g_m(v_{m+1}) - g_m(v_m)\|. \end{aligned}$$

In order to estimate the last summand on the right hand side, note that the value $S_m g_m(v_{m+1})$ is uniquely determined by $g_m|_{E(M_m^k) \cap B(v_{m+1}, 2^{-m}R)}$, where R is the constant from Lemma 2.5. With the constant C_2 of the same lemma it follows that $d(v_m, v_{m+1}) \leq \frac{3}{2} C_2 2^{-m}$. Consequently, $\max\{d(v_m, y) : y \in E(M_m^k) \cap B(v_{m+1}, 2^{-m}R)\} \leq \frac{3}{2} C_2 2^{-m} + 2^{-m}R$. The left hand inequality in Lemma 2.5*(i)* now implies that the number of faces in $\mathcal{F}_m$ not disjoint to $B(v_m, (\frac{3}{2} C_2 + R) 2^{-m})$ is bounded by $D \in \mathbb{N}$, where D is independent of m or v_m. With $B^* := B(v_{m+1}, 2^{-m}R)$, we can write $S_m g_m(v_{m+1}) = \sum_{q \in E(M_m^k) \cap B^*} \alpha_q g_m(q)$ with $\sum_{q \in E(M_m^k) \cap B^*} \alpha_q = 1$ and $\sum_{q \in E(M_m^k) \cap B^*} |\alpha_q| \leq \|A_m\|$. We obtain

$$\begin{aligned} \|S_m g_m(v_{m+1}) - g_m(v_m)\| &= \left\| \sum\nolimits_{q \in E(M_m^k) \cap B^*} \alpha_q (g_m(q) - g_m(v_m)) \right\| \\ &\leq \sum\nolimits_{q \in E(M_m^k) \cap B^*} |\alpha_q| \cdot \max\nolimits_{q \in E(M_m^k) \cap B^*} \|g_m(q) - g_m(v_m)\| \leq \|S_m\| D \mathcal{D}(g_m). \end{aligned}$$

Altogether, it follows that

$$\|I_{m+1}S_m g_m - I_m g_m\| \leq \mathcal{D}(S_m g_m) + (\|S_m\|D + 1)\mathcal{D}(g_m).$$

Equipped with this inequality, we estimate, for $n > l$,

$$\begin{aligned}\|I_{n+1}S_{n,l}p_l - I_n S_{n-1,l}p_l\|_\infty \leq & \mathcal{D}_{\mathcal{K}_{n+1}} S_{n,l}p_l + (\|S_n\|D + 1)\mathcal{D}_{\mathcal{K}_n}(S_{n-1,l}p_l) \\ \leq & C\gamma^{n-l}(\|S_n\|D + 2)\mathcal{D}(p_l).\end{aligned}$$

For $n'' \geq n' \geq n \geq l$ we make use of the geometric series and get

$$\|I_{n''+1}S_{n'',l}p_l - I_{n'}S_{n'-1,l}p_l\|_\infty \leq C(\sup_{n\in\mathbb{N}_0}\|S_n\|D + 2)\gamma^{n-l}\frac{1}{1-\gamma}\mathcal{D}(p_l). \tag{2.6}$$

Thus $\{I_n S_{n-1,l}p_l\}_{n>l}$ is Cauchy in the space of bounded continuous functions. Since these functions are uniformly continuous, so is the limit, called f for the moment. Now, $\|f|_{E(M_n^k)} - S_{n-1,l}p_l\|_\infty \leq \|f - I_n S_{n-1,l}p_l\| \to 0$ for $n \to \infty$. Thus f equals $S_{\infty,l}p_l$. Letting $n' = l$ in (2.6) yields the estimate

$$\|f - I_l p_l\| = \lim_{n''\to\infty}\|I_{n''+1}S_{n'',l}p_l - I_l p_l\| \leq \frac{1}{1-\gamma}(\sup_{n\in\mathbb{N}_0}\|S_n\|D + 2)\mathcal{D}(p_l).$$

This proves the last statement of the proposition. □

Proposition 2.7. *Let S be a standard scheme acting on data defined on $E(M_n^k)$, and suppose* (2.5) *holds true. Let furthermore T and M'' be as in Theorem 2.4, with M_n^k replaced by its image under E. Assume also that S and T fulfill a local proximity condition w.r.t. some $P_{M,\delta}$. Then there is $\delta'' > 0$ such that for any input $g_0 \in P_{M'',\delta''}$ on level* 0*, $T_{l-1,0}g_0$ $(l \in \mathbb{N})$ is defined and*

$$\mathcal{D}(T_{l-1,0}g_0) \leq 2C\gamma^l\mathcal{D}(g_0), \tag{2.7}$$

with the same C and γ as in (2.5)*. For such g_0, $\{T_{l-1,0}g_0\}_{l\in\mathbb{N}}$ converges and the sequence $\{I_l T_{l-1,0}g_0\}_{l\in\mathbb{N}}$ converges to the same limit in $C_u(\mathbb{R}^2, \mathbb{R}^d)$.*

Proof. We denote the constant of (2.1) by F, and use M' and δ' from Theorem 2.4. C_B is the constant from Proposition 2.6. We set

$$\delta'' := \min\left\{\frac{(1-\gamma)\gamma}{8FC^2}, \frac{\delta}{2C}, \frac{1-\gamma}{4C_B C}\delta', \left(\frac{1-\gamma^2}{8FC^2}\delta'\right)^{\frac{1}{2}}\right\}.$$

We intend to use Lemma 2.2 and induction on l. We start with $l = 1$. Since $g_0 \in l^\infty(E(M_0^k), M'')$, $M'' \subset M$, and $\mathcal{D}(g_0) < \delta$, data g_0 lie in the domain of T_0. Let $g_1 = T_0 g_0$. Now (2.1) ensures (2.2) with $C' = \frac{F}{\gamma}$, and since $\mathcal{D}(g_0) \leq \frac{(1-\gamma)\gamma}{8FC^2}$, (2.3) is fulfilled. Hence $\mathcal{D}(T_0 g_0) \leq 2C\gamma\mathcal{D}(g_0) < \delta$. Now

$$\begin{aligned}\|I_1 T_0 g_0 - I_0 g_0\| &\leq \|I_1 T_0 g_0 - I_1 S_0 g_0\| + \|I_1 S_0 g_0 - I_0 g_0\| \\ &\leq \|T_0 g_0 - S_0 g_0\| + C_B\mathcal{D}(g_0) \leq F\mathcal{D}(g_0)^2 + C_B\mathcal{D}(g_0) \leq \delta'.\end{aligned}$$

It follows that $T_0 g_0$ takes its values in M, and that $T_0 g_0$ is in the domain of T_1.

We now perform the induction step. Assume that $g_m = T_{m-1,0} g_0$ is defined, that $T_{m-1,0} g_0$ takes its values in M, and that $T_{m-1,0} g_0$ is in the domain of T_m, for $0 \le m \le l$. Then (2.1) ensures (2.2), again with $C' = \frac{F}{\gamma}$. Lemma 2.2 yields $\mathcal{D}(T_{l,0} g_0) \le 2C\gamma^{l+1}\mathcal{D}(g_0) < \delta$. Again,

$$\begin{aligned}
&\|I_{l+1} T_{l,0} g_0 - I_0 g_0\| \\
&\quad \le \sum\nolimits_{m=0}^{l} \|I_{m+1} T_{m,0} g_0 - I_{m+1} S_m T_{m-1,0} g_0\| + \|I_{m+1} S_m T_{m-1,0} g_0 - S_m T_{m-1,0} g_0\| \\
&\quad \le F \sum\nolimits_{m=0}^{l} \mathcal{D}(T_{m-1,0} g_0)^2 + C_B \sum\nolimits_{m=0}^{l} \mathcal{D}(T_{m-1,0} g_0) \\
&\quad \le 4FC^2 \left(\sum\nolimits_{m=0}^{\infty} \gamma^{2m}\right) \mathcal{D}(g_0)^2 + 2C_B C \sum\nolimits_{m=0}^{\infty} \gamma^m \mathcal{D}(g_0) \\
&\quad \le \frac{4FC^2}{1-\gamma^2}\mathcal{D}(g_0)^2 + \frac{2C_B C}{1-\gamma}\mathcal{D}(g_0) < \delta'.
\end{aligned}$$

Thus $T_{l,0} g_0$ takes its values in M, and is in the domain of T_{l+1}. This completes the induction step.

For the convergence statement, assume that $l'' \ge l' \ge l$. Then we have

$$\begin{aligned}
\|I_{l''+1} T_{l'',0} g_0 - I_{l'+1} T_{l',0} g_0\| &\le \frac{4FC^2}{1-\gamma^2}\mathcal{D}(T_{l-1,0} g_0)^2 + \frac{2C_B C}{1-\gamma}\mathcal{D}(T_{l-1,0} g_0) \\
&\le \frac{16FC^4}{1-\gamma^2}\gamma^{2l}\mathcal{D}(g_0)^2 + \frac{4C_B C^2}{1-\gamma}\gamma^l \mathcal{D}(g_0).
\end{aligned}$$

Since the right hand side approaches 0 as $l \to \infty$, the sequence $\{I_l T_{l-1,0} g_0\}_{l \in \mathbb{N}}$ is Cauchy in $C_u(\mathbb{R}^2, \mathbb{R}^d)$ and therefore convergent. □

Lemma 2.8. *Under the same assumptions as in Proposition 2.7, the sequence $S_{\infty,l} T_{l-1,0} g_0$ converges to $T_{\infty,0} g_0$ in $C_u(\mathbb{R}^2, \mathbb{R}^d)$ as $l \to \infty$.*

Proof. For $\varepsilon > 0$, choose $L \in \mathbb{N}$ such that for all $l \ge L$, $\|T_{\infty,0} g_0 - I_l T_{l-1,0} g_0\| < \frac{\varepsilon}{2}$. By Proposition 2.6 there is $C_I > 0$ such that

$$\|S_{\infty,l} T_{l-1,0} g_0 - I_l T_{l-1,0} g_0\| \le C_I \mathcal{D}(T_{l-1,0} g_0) \le 2C_I C \gamma^l \mathcal{D}(g_0).$$

Now choose $L_0 > L$ such that $2C_I C \gamma^{L_0} < \frac{\varepsilon}{2}$. Then for all $l \ge L_0$, $\|T_{\infty,0} g_0 - S_{\infty,l} T_{l-1,0} g_0\| < \varepsilon$. □

We have collected sufficient results to give the **proof of Theorem 2.4**.

Proof. It remains to show (2.5) for the operators $\{S_n\}_{n \in \mathbb{N}_0}$, since then Theorem 2.4 immediately follows from Proposition 2.7. We consider the rings $\{P_i\}_{i=-1}^{\infty}$, where we let $P_{-1} := \{(x, y, j) \in \tilde{P} : \max(x, y) \ge r\}$. Lemma 2.3 yields $C_2 \ge 1$ and $\gamma'' \in (0, 1)$ such that

$$\sup_{v,w \in \operatorname{ctrl}^i(P_i)} \|S_{i-1,0} p_0(v) - S_{i-1,0} p_0(w)\|_{\mathbb{R}^d} \le C_2 (\gamma'')^i \sup_{v,w \in \operatorname{ctrl}^0(P_0)} \|p_0(v) - p_0(w)\|_{\mathbb{R}^d}.$$

Since the sets $\mathrm{ctrl}^i(P_i)$ are finite, the triangle inequality yields $C_3 > 0$ such that $\mathcal{D}_{\mathrm{ctrl}^i(P_i)}(S_{i-1,0}p_0) \leq C_4(\gamma'')^i\, \mathcal{D}_{\mathrm{ctrl}^0(P_0)}(p_0)$. Now fix $n \in \mathbb{N}$. Then for any $g_n \in l^\infty(M_n^k, \mathbb{R}^d)$, $\mathcal{D}(g_n) = \sup_{-1\leq i\leq n} \mathcal{D}_{\mathrm{ctrl}^n(P_i)}(g_n)$. Data $S_{n-1,0}p_0$ restricted to $\mathrm{ctrl}_n(P_i^j)$ is obtained from $S_{i-1,0}p_0$ on $\mathrm{ctrl}_i(P_i^j)$, by subdivision w.r.t. a regular mesh connectivity. Therefore, Proposition 1.1 and the triangle inequality yield constants $C_1 \geq 1$ and $\gamma' \in (0,1)$ such that $\mathcal{D}_{\mathrm{ctrl}^n(P_i^j)}(S_{n-1,0}p_0) \leq C_1(\gamma')^{n-i}\, \mathcal{D}_{\mathrm{ctrl}^i(P_i)}(S_{i-1,0}p_0)$. Now, for $i \geq 0$,

$$\begin{aligned}\mathcal{D}_{\mathrm{ctrl}^n(P_i^j)}(S_{n-1,0}p_0) &\leq C_1(\gamma')^{n-i}\mathcal{D}_{\mathrm{ctrl}^i(P_i)}(S_{i-1,0}p_0)\\ &\leq C_1C_3(\gamma')^{n-i}(\gamma'')^i\mathcal{D}_{\mathrm{ctrl}^0(P_0)}(p_0) \leq C\max(\gamma',\gamma'')^n\mathcal{D}(p_0)\end{aligned}$$

with $C := C_1C_3$. For $i = -1$, we have the inequality

$$\mathcal{D}_{\mathrm{ctrl}^n(P_{-1})}(S_{n-1,0}p_0) \leq C_1(\gamma')^n\mathcal{D}_{\mathrm{ctrl}^0(P_{-1})}(p_0).$$

This completes the proof. □

Remark 2.9. Proposition 2.6 and Proposition 2.7 are actually valid in a more general setting: If the requirements of Lemma 2.5, where we can replace the 2 by $m > 1$, are fulfilled for a sequence of arbitrary operators S_n, point sets, and face sets, then Proposition 2.6 is still valid, and works as a convergence proof. Subsequently, Proposition 2.7 carries over to this more general setting with the same proof.

2.2 Smoothness Analysis

So far we have shown convergence for a nonlinear scheme T, which is in proximity to a standard scheme S. In this section we analyse the C^1 smoothness of $T_{\infty,0}p_0 \circ (\chi|_P)^{-1}$, where χ denotes Reif's characteristic parametrisation over the relevant set $P \subset \tilde{P}$.

More precisely, we reconsider the sequence $S_{\infty,n}T_{n-1,0}p_0$ which converges to $T_{\infty,0}p_0$ in $C(P,\mathbb{R}^d)$ by Lemma 2.8. We are going to show that this convergence is true even in the space $C^1(\chi(P),\mathbb{R}^d)$. The main statement is the following:

Theorem 2.10. *Let S be a standard subdivision scheme, and assume that S and T fulfill a local proximity condition w.r.t. $P_{M,\delta}$. Then for $p_0 \in P_{M'',\delta''}$ (see Proposition 2.7), the function $T_{\infty,0}p_0 \circ \chi^{-1}$ is continuously differentiable, where $T_{\infty,0}p_0 :\ P \to M$ is the limit function of T, and $\chi : P \to \mathbb{R}^2$ is the characteristic map.*

Notice that if T converges, data eventually get dense enough. So for showing smoothness, a 'dense enough' assumption is no restriction. In order to show Theorem 2.10 we first prove a series of lemmas.

Lemma 2.11. *Let T be in proximity to a standard scheme S w.r.t. $P_{M,\delta}$, and let $p_0 \in P_{M'',\delta''}$. Then there is $C_1 \geq 1$ such that for $i \geq l$ and $j \in \mathbb{Z}/k\mathbb{Z}$,*

$$\mathcal{D}_{\mathrm{ctrl}^i(P_l^j)}(T_{i-1,0}p_0) \leq C_1 2^{-i+l}\mathcal{D}_{\mathrm{ctrl}^l(P_l^j)}(T_{l-1,0}p_0). \tag{2.8}$$

Furthermore there is $C_2 \geq 1$ such that for $l \in \mathbb{N}$

$$\mathcal{D}_{\mathrm{ctrl}^l(P_l)}(T_{l-1,0}p_0) \leq C_2\lambda^l\mathcal{D}_{\mathrm{ctrl}^0(P_0)}(p_0), \tag{2.9}$$

where λ is the subdominant eigenvalue of the subdivision matrix A.

Proof. We begin with the first statement. Note that $T_{k,0}p_0$ is defined for any $k \in \mathbb{N}$. For $i \geq l$, $S_{i-1,l}T_{l-1,0}p_0|_{\mathrm{ctrl}^i(P_l^j)}$ is determined by $T_{l-1,0}p_0|_{\mathrm{ctrl}^l(P_l^j)}$ by means of subdivision w.r.t. regular connectivity. Proposition 1.1 and the triangle inequality yield $C' > 0$ such that

$$\mathcal{D}_{\mathrm{ctrl}^i(P_l^j)}(S_{i-1,l}T_{l-1,0}p_0) \leq C' 2^{-i+l}\mathcal{D}_{\mathrm{ctrl}^l(P_l^j)}(T_{l-1,0}p_0).$$

This constant C' is independent of i, j, l and data p_0. We apply Lemma 2.2 with $\gamma = 1/2$ to the sets $\{\mathrm{ctrl}^i(P_l^j)\}_{i \geq l}$. In Lemma 2.2, we start on subdivision level l instead of level 0. The locality of the proximity condition guarantees that the conditions of Lemma 2.2 are met. We conclude that (2.8) holds true.

We show the second statement. From Lemma 2.3 we get $C' > 0$ such that

$$\mathcal{D}_{\mathrm{ctrl}^l(P_l)}(S_{l-1,0}p_0) \leq C'\lambda^l \mathcal{D}_{\mathrm{ctrl}^0(P_0)}(p_0).$$

Then we apply Lemma 2.2 for $\{\mathrm{ctrl}^l(P_l)\}_{l \in \mathbb{N}_0}$. Again, by the locality of the proximity condition the assumptions of Lemma 2.2 are fulfilled, and (2.9) follows. □

Proposition 2.12. *Let a standard scheme S and a (nonlinear) scheme T fulfill a local proximity condition w.r.t. $P_{M,\delta}$. Let furthermore $\chi : P \to \mathbb{R}^2$ be the characteristic map, and $p_0 \in P_{M'',\delta''}$. Then $S_{\infty,i}T_{i-1,0}g_0 \circ \chi^{-1} \in C^1(\chi(P), M)$. In addition, there is $C \geq 1$ such that for $i \geq n$, and $j \in \mathbb{Z}/k\mathbb{Z}$,*

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0) \circ \chi^{-1}|_{\chi(P_n^j)}\|_{C^1(\chi(P_n^j),\mathbb{R}^d)} \leq C\gamma^i \mathcal{D}_{\mathrm{ctrl}^0(P_0)}(p_0)^2, \tag{2.10}$$

where $\gamma := \max(2^{-1}, \lambda)$, and λ is the subdominant eigenvalue of the subdivision matrix A.

Proof. By Theorem 1.2, $S_{\infty,0}p_0 \circ \chi^{-1} \in C^1(\chi(P), M)$. By the scaling property of the characteristic map, i.e., $\chi(\cdot/2^m) = \lambda^m\chi$, and since S produces C^1-limits on regular connectivities, $S_{\infty,i}T_{i-1,0}p_0 \circ \chi^{-1}$ is C^1.

In order to prove (2.10) we first show that there is $C_3 > 0$, which is independent of i, j, and n, such that

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0)|_{P_n^j}\|_{C^1(P_n^j,\mathbb{R}^d)} \leq C_3 2^i \|(T_i - S_i)T_{i-1,0}p_0\|_\infty. \tag{2.11}$$

On a regular connectivity, the operator S from Proposition 1.1 commutes with translation and has finite support. Hence $S_{\infty,i}$ is a bounded linear operator from $l^\infty(\mathrm{ctrl}^i(P_n^j), \mathbb{R}^d)$ to $C^1(P_n^j, \mathbb{R}^d)$. Scaling a grid by two at most doubles the C^1-norm, so for any $f_i \in l^\infty(\mathrm{ctrl}^i(P_n^j), \mathbb{R}^d)$, we get

$$\|S_{\infty,i}f_i\|_{C^1(P_n^j,\mathbb{R}^d)} \leq 2^i \|S_{\infty,0}\|_{l^\infty \to C^1} \|f_i\|_\infty.$$

This implies (2.11).

Since χ is a diffeomorphism in a neighbourhood of P_0^j, all $h \in C^1(P_0^j, \mathbb{R}^d)$ obey the inequality $\|h \circ \chi^{-1}|_{\chi(P_n^j)}\|_{C^1} \leq D\|h\|_{C^1}$ for some $D > 0$, which is independent of h. Using the scaling relation $\chi(\cdot/2^n) = \lambda^n\chi$ again, yields $C_4 > 0$ which is independent of i, j, and n such that

$$\begin{aligned}&\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0) \circ \chi^{-1}|_{\chi(P_n^j)}\|_{C^1(\chi(P_n^j),\mathbb{R}^d)} \\ &\qquad \leq 2^{i-n}\lambda^{-n}C_4\|(T_i - S_i)T_{i-1,0}p_0|_{\mathrm{ctrl}^{i+1}(P_n^j)}\|_\infty.\end{aligned}$$

We now use the proximity condition (2.1), and obtain that there is $C_5 > 0$ such that

$$\begin{aligned}\|(T_i - S_i)T_{i-1,0}p_0|_{\mathrm{ctrl}^{i+1}(P_n^j)}\|_\infty &\leq C_5 \left(\mathcal{D}_{\mathrm{ctrl}^i(P_n^j)}(T_{i-1,0}p_0)\right)^2 \\ &\leq C_5 C_2 C_1 \lambda^{2n} 2^{-2i+2n} \mathcal{D}_{\mathrm{ctrl}^0(P_0)}(p_0)^2,\end{aligned}$$

with the constants C_1 and C_2 from Lemma 2.11. Altogether,

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0)|_{P_n^j}\|_{C^1(P_n^j,\mathbb{R}^d)} \leq C_1 C_2 C_4 C_5 \lambda^n 2^{-i+n} \mathcal{D}_{\mathrm{ctrl}^0(P_0)}(p_0)^2.$$

This completes the proof. □

Proposition 2.13. *Let T be in proximity to a standard scheme S w.r.t. $P_{M,\delta}$, and let $p_0 \in P_{M'',\delta''}$. Then $\{S_{\infty,i+1}T_{i,0}p_0\}_{i\in\mathbb{N}_0}$ is a Cauchy sequence in $C^1(\chi(P),\mathbb{R}^d)$.*

Proof. The linear operators

$$L_i :\ l^\infty(\mathrm{ctrl}^i(P_i),\mathbb{R}^d) \to C^1(\chi\left(\cup_{m=i}^\infty P_m \cup \{0\}\right),\mathbb{R}^d),$$

assigning the limit function of subdivision over the characteristic parametrisation to data on $\mathrm{ctrl}^i(P_i)$, are bounded, since they operate on finite dimensional space. We consider, for $i,k \in \mathbb{N}_0$, the isometric isomorphism

$$\begin{aligned}&V_{i,k} :\ l^\infty(\mathrm{ctrl}^i(P_i),\mathbb{R}^d) \to l^\infty(\mathrm{ctrl}^k(P_k),\mathbb{R}^d),\\ &V_{i,k}p_i(x) = p_i(2^{-i+k}x).\end{aligned}$$

We have $V_{i,k}\circ L_i = L_k \circ V_{i,k}$. Now the scaling property of the rings of the characteristic map $\{\chi(P_i)\}_{i\in\mathbb{N}_0}$ for any $i \in \mathbb{N}_0$ and any $p_i \in l^\infty(\mathrm{ctrl}^i(P_i),\mathbb{R}^d)$ yields the estimate $\|L_i p_i\|_{C^1} \leq \|L_0\|\lambda^{-i}\|p_i\|_\infty$, where λ again denotes the subdominant eigenvalue of the subdivision matrix. Then there is $C_3 > 0$ from the proximity condition (2.1) such that

$$\begin{aligned}&\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0)\circ\chi^{-1}|_{\chi\left(\cup_{m=i}^\infty P_m \cup\{0\},\mathbb{R}^d\right)}\|_{C^1} \\ &\qquad = \|L_{i+1}(T_i - S_i)T_{i-0,0}p_0\|_{C^1} \leq \lambda^{-i}\|L_0\|C_3\mathcal{D}_{\mathrm{ctrl}^i(P_i)}(T_{i-1,0}p_0)^2 \\ &\qquad \leq C'\lambda^{-i}\lambda^{2i}\mathcal{D}_{\mathrm{ctrl}^0(P_0)}(p_0)^2 \leq C'\lambda^i\mathcal{D}_{\mathrm{ctrl}^0(P_0)}(p_0)^2,\end{aligned} \tag{2.12}$$

where $C' = C_1C_2C_3\|L_0\|$ with the constants C_1 and C_2 of Lemma 2.11. We know from Proposition 2.12 that for any $i \in \mathbb{N}_0$, the limit function $S_{\infty,i+1}T_{i,0}p_0\circ\chi^{-1} \in C^1(\chi(P),\mathbb{R}^d)$. We use both (2.10) and (2.12) and see that

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i,0}p_0)\circ\chi^{-1}|_{\chi(P)}\|_{C^1} \leq C\gamma^i\mathcal{D}_{\mathrm{ctrl}^0(P_0)}(p_0)^2$$

for some $C > 0$ and $\gamma := \max(2^{-1},\lambda)$. This implies

$$\|S_{\infty,k}T_{k-1,0}p_0 - S_{\infty,l}T_{l,0}p_0\circ\chi^{-1}|_{\chi(P)}\|_{C^1} \leq C\gamma^{\min(k,l)}\frac{1}{1-\gamma}\mathcal{D}_{\mathrm{ctrl}^0(P_0)}(p_0)^2, \tag{2.13}$$

which completes the proof. □

Finally we are able to settle the **proof of Theorem 2.10**.

Proof. By Lemma 2.8, $S_{\infty,n}T_{n-1,0}p_0$ converges to $T_{\infty,0}p_0$ on P in the sup norm. Since this sequence is Cauchy on $\chi(P)$ with respect to the C^1 norm, its limit $T_{\infty,0}p_0 \circ \chi^{-1}$ must be continuously differentiable. □

We can add a condition which guarantees that $T_{\infty,0}p_0(P)$ locally is a submanifold around the extraordinary point $T_{\infty,0}p_0(0)$. Note that the statement below is not as strong as the respective statement in the linear case.

Corollary 2.14. *Let a standard scheme S and a (nonlinear) scheme T be in proximity w.r.t. $P_{M,\delta}$, and let $p_0 \in P_{M'',\delta''}$ (see Proposition 2.7). Assume that the Jacobian $J_0(S_{\infty,0}p_0 \circ \chi^{-1})$ in the extraordinary point 0 of the limit function of linear subdivision using S fulfills $\|J_0(S_{\infty,0}p_0 \circ \chi^{-1})(x)\|_\infty \geq \xi\|x\|_\infty$ for some $\xi > 0$. Assume further that*

$$\mathcal{D}_{\mathrm{ctrl}^0(P)}(p_0) < (\xi(1-\gamma)/C)^{\frac{1}{2}},$$

where C is the constant from (2.13), $\gamma = \max(2^{-1}, \lambda)$, and λ is the subdominant eigenvalue of the subdivision matrix A. Then also the nonlinear scheme T produces a 2-dimensional manifold locally around the extraordinary point.

Proof. From (2.13) it follows that

$$\|(S_{\infty,0}p_0 - T_{\infty,0}p_0) \circ \chi^{-1}|_{\chi(P)}\|_{C^1} \leq C(1-\gamma)^{-1}\mathcal{D}_{\mathrm{ctrl}^0(P)}(p_0)^2.$$

Thus, for any $x \in \mathbb{R}^2$,

$$\begin{aligned}\|J_0(T_{\infty,0}p_0 \circ \chi^{-1})(x)\| &\geq \|J_0(S_{\infty,0}p_0 \circ \chi^{-1})(x)\| - \|S_{\infty,0}p_0 - T_{\infty,0}p_0 \circ \chi^{-1}\|_{C^1} \\ &\geq \xi - C(1-\gamma)^{-1}\mathcal{D}(p_0)^2 > 0.\end{aligned}$$

This shows that the Jacobian is regular in 0, which completes the proof. □

We still owe the **proof of Theorem 1.4**.

Proof. By Theorem 2.4 and Theorem 2.10, convergence and smoothness are ensured, if a local proximity condition holds. Although Wallner and Dyn's proximity inequality in [15] is slightly weaker, they actually prove our local proximity condition for the projection analogue in [15], Lemma 7, and for the geodesic analogue in [15], Lemma 5. A proof which works for the log-exp analogue is the proof of [4], Proposition 7.2. Then the local proximity condition (1.4) for the intrinsic mean analogue is a consequence of its interpretation as log-exp analogue with special base points. □

3 An Application

As an application we show how subdivision in the geometric setting can be used to generate manifold-valued smooth functions on smooth two-dimensional manifolds. To that end, we consider two meshes with the same connectivity. Let us assume the first mesh has its values in the smooth manifold N. Let the second mesh 'cover' a smooth 2-manifold M, and assume that the positioning function is 1-1. Then we have a map from the positions in N to that in M. Now, let S be a standard scheme, T be an analogue acting in M, and

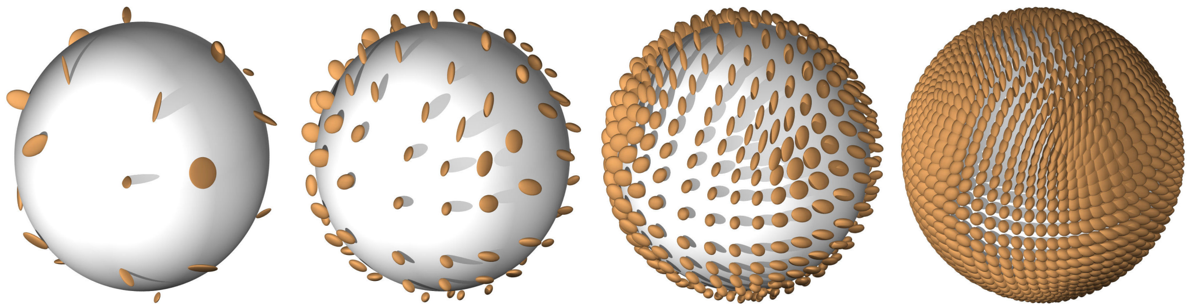

Figure 2: Initial rounds of subdivision for a subdivided cube connectivity.

T' an analogue acting in N. Iterated application of both T and T' simultaneously yields a sequence of mappings, defined in discrete subsets of M, with values in N. The first steps of this process are visualized in Figure 2 and Figure 3. Here we used the projection analogue on spheres on the one hand, and intrinsic mean subdivision in the Riemannian manifold of positive matrices on the other hand. The theoretical fundament is given by the following corollary, formulated near extraordinary points of valence k.

Corollary 3.1. *Let T and T' be analogues of the same standard scheme S, and let input data $p_0 : M_0^k \to M$ and $p'_0 : M_0^k \to M$ be dense enough. If $T_{\infty,0}p_0 \circ \chi^{-1} : \chi(P) \to M$ is injective and regular, then*

$$T'_{\infty,0}p'_0 \circ (T_{\infty,0}p_0)^{-1} : T_{\infty,0}p_0(P) \to T'_{\infty,0}p'_0(P) \tag{3.1}$$

is a C^1 mapping.

Note that Corollary 2.14 gives a sufficient condition for regularity near the extraordinary point. Then, at least in a small neighbourhood, we also have injectivity.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & \chi(P) \subset \mathbb{R}^2 \arrow[dr, "C^1"] & \\
M \supset T_{\infty,0}p_0(P) \arrow[ur, "C^1"] & P \arrow[l, "T_{\infty,0}p_0"'] \arrow[r, "T'_{\infty,0}p'_0"] \arrow[u, "\chi"] & T'_{\infty,0}p'_0(P) \subset N
\end{tikzcd}$$

This means that $T'_{\infty,0}p'_0 \circ (T_{\infty,0}p_0)^{-1} = T'_{\infty,0}p'_0 \circ \chi^{-1} \circ (T_{\infty,0}p_0 \circ \chi^{-1})^{-1}$, where χ is the characteristic map. Now the statement follows from Theorem 2.10. □

4 Conclusion and Future Research

We have shown that geometric, nonlinear, analogues of primal or dual quad based linear subdivision schemes for 2D irregular meshes converge, provided input data are dense enough and certain technical conditions are met. Under these conditions, we obtain

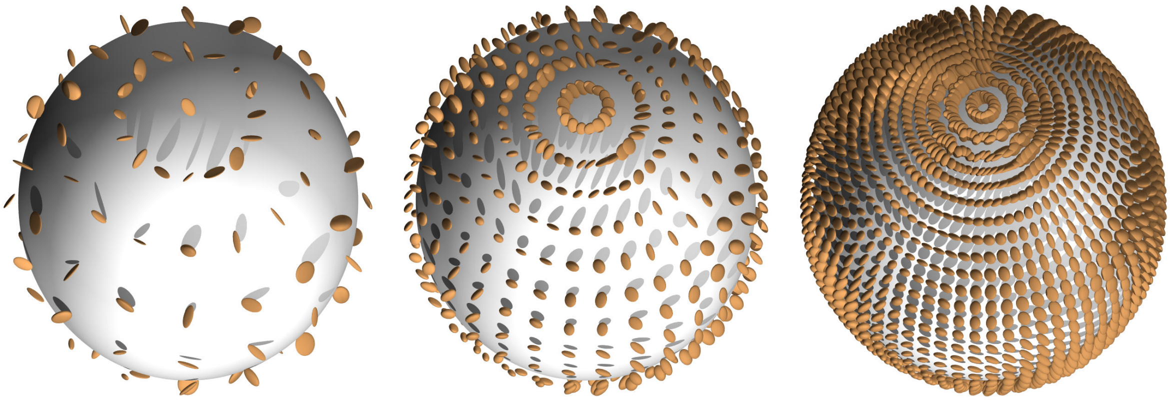

Figure 3: Two rounds of subdivision analogous to the Doo-Sabin scheme using projection and intrinsic means. One can cleary observe the oscillation near the valence 16 extraordinary face, especially in the data position. Also note that 8 valence 3 faces are nearby.

that these schemes produce C^1 limit functions over a domain constructed from Reif's characteristic parametrisation. As an application we have described a quite general way of constructing smooth manifold valued functions on 2D manifolds from discrete data.

The present paper only treats quadrilateral meshes. We would like to mention that the case of primal triangle subdivision is analogous to the case of primal quad meshes. We should also point out that our proof works for dilation factors greater than two as well.

One topic of future research is the convergence and smoothness analysis of nonlinear subdivision schemes based on different topological refinement rules, e.g. the quincunx and $\sqrt{3}$ schemes, where also the regular grid case has not been analysed yet. It would also be very interesting to investigate nonlinear wavelet-type transforms defined on surfaces which do not allow for a covering by regular meshes. In the regular mesh case, results on such transforms have been recently obtained in [5]. A main point in the irregular case seems to be the correct choice of sample points, and we plan to employ subdivision for that purpose.

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