Lipschitz Spaces with respect to Jacobi Translation

Andreas Weinmann *[1] and **Rupert Lasser** **[2]

[1] Institute of Geometry, TU Graz, Kopernikusgasse 24, 8010 Graz, Austria.
[2] Institue of Biomathematics and Biometry, HelmholtzZentrum München, Ingolstädter Landstraße 1, 85764 Neuherberg, Germany.

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The Jacobi polynomials induce a translation operator on function spaces on the interval $[-1, 1]$. For any homogeneous Banach space B w.r.t. this translation, we can study the according little and big Lipschitz spaces, $\mathrm{lip}_B(\lambda)$ and $\mathrm{Lip}_B(\lambda)$, respectively. The big Lipschitz spaces are not homogeneous themselves.

Therefore we introduce semihomogeneous Banach spaces w.r.t. Jacobi translation, of which the big Lipschitz spaces are particular examples. We study the relation between semihomogeneous Banach spaces and their homogeneous counterparts. We give a characterisation of Lipschitz spaces in terms of intermediate spaces. Our main result is that, for an arbitrary homogeneous Banach space B, the bidual of the little Lipschitz space $\mathrm{lip}_B(\lambda)$ is the corresponding big one, namely $\mathrm{Lip}_B(\lambda)$.

1 Introduction

It is natural to generalize the concept of a Lipschitz function f on the real line, i.e., f obeys $|f(y) - f(x)| < C|x - y|^\lambda$ $(0 < \lambda < 1)$, to general metric spaces. Here $|x - y|^\lambda$ is replaced by a distance $d(x, y)$. A lot of work has been done on spaces of Lipschitz functions on metric spaces, see e.g. Weaver [22].

On the real line, a Lipschitz function f is also characterized by $\|T_h f - f\|_\infty \le C|h|^\lambda$, with the ordinary translation operators T_h. This expression also makes sense if we replace the sup-norm by the norm of some homogeneous Banach space B, e.g. $B = L^1(\mathbb{R})$. Then the Lipschitz space w.r.t. B consists of the elements in B with $\|T_h f - f\|_B \le C|h|^\lambda$. By replacing the (induced) ordinary translation on functions by a generalized translation we obtain a different generalization of the concept of Lipschitz spaces.

In this paper, we consider the interval $[-1, 1]$ and a (generalized) translation induced by the Jacobi polynomials. We investigate Lipschitz spaces w.r.t. arbitrary homogeneous Banach spaces.

(Semi-)homogeneous Banach spaces and Segal algebras on locally compact groups are well understood, see e.g. the book [21]. In order to generalize these concepts to $S = [-1, 1]$, on which no group structure is imposed, one first of all needs a notion of translation on the according L^1-space. We stick to the interval equipped with its Borel σ-Algebra and the measure $\pi^{(\alpha,\beta)}$, whose density w.r.t. Lebesgue measure λ is given by $c_{\alpha,\beta}(1 - x)^\alpha(1 + x)^\beta$. Here the normalisation constant $c_{\alpha,\beta} = 2^{-\alpha-\beta-1}\,\Gamma(\alpha + \beta + 2)\,\Gamma(\alpha + 1)^{-1}\,\Gamma(\beta + 1)^{-1}$ is chosen such that $\pi^{(\alpha,\beta)}(S) = 1$. The Jacobi polynomials $R_n^{(\alpha,\beta)}$, $n \in \mathbb{N}_0$, are orthogonal w.r.t. $\pi^{(\alpha,\beta)}$. We choose the normalization $R_n^{(\alpha,\beta)}(1) = 1$, and assume throughout that $(\alpha, \beta) \in J$, where $J = \{(\alpha, \beta) : \alpha \ge \beta \ge -1/2 \vee (\alpha \ge \beta > -1 \wedge \alpha + \beta \ge 0)\}$. In that case we have for all $x, y \in S$ the positive linearization

$$R_n^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y) = \int_{-1}^{1} R_n^{(\alpha,\beta)}(z)\, d\pi_{(x,y)}^{(\alpha,\beta)}(z), \tag{1.1}$$

where $\pi_{(x,y)}^{(\alpha,\beta)}$ is a probability measure. This important result is due to Gasper, see [11]. We frequently drop the parameters (α, β) and think of them as fixed and being contained in J. Details on Jacobi polynomials can be found in [19].

* e-mail: andreas.weinmann@tugraz.at, Phone: +43 316 873 8446, Fax: +43 316 873 8446
** e-mail: lasser@helmholtz-muenchen.de, Phone: +49 89 3187 4159, Fax: +49 89 3187 3369

For any $x \in S$, a (generalized) translation operator on $L^1(S,\pi)$ is given by

$$T_x f(y) = \int_{-1}^{1} f(z)\, d\pi_{(x,y)}(z). \tag{1.2}$$

For all $x \in S$, we have that $\|T_x f\|_1 \leq \|f\|_1$, and furthermore the translation $S \to L^1(S,\pi)$, $x \to T_x f$ is continuous. These statements can be proved in the framework of hypergroups; we refer to [4]. In fact, S bears a hypergroup structure induced by the $\pi_{(x,y)}$, see [14]. So we have the tools of harmonic analysis on commutative compact hypergroups at our disposal.

(Semi-)homogeneous Banach spaces w.r.t. Jacobi translation can now be defined as follows.

Definition 1.1 Let T be a linear subspace of $L^1(S,\pi)$ which becomes a Banach space with a norm $\|\ \|_T$. T is called semihomogeneous, if

(B1) T contains all Jacobi polynomials R_n $(n \in \mathbb{N}_0)$,
(B2) for all $f \in T$, $\|f\|_{L^1} \leq \|f\|_T$,
(B3) for $f \in T$ and $x \in [-1,1]$, $T_x f \in T$ and $\|T_x f\|_T \leq \|f\|_T$.

T is called homogeneous, if in addition,

(B4) for fixed $f \in T$, the translation $S \to T,\ x \mapsto T_x f$, is continuous.

Homogeneous Banach spaces with respect to Jacobi polynomials were introduced in [10]. Examples can also be found there. Semihomogeneous Banach spaces with respect to commutative locally compact groups were e.g. treated in [21].

Using the (generalized) translation, the convolution on $L^1(S,\pi)$ is given by

$$(f * g)(x) := \int_{-1}^{1} f(z) T_x g(z)\, d\pi(z). \tag{1.3}$$

We have $f * g = g * f$, and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. It follows that $L^1(S,\pi)$ is a commutative Banach algebra with this convolution as multiplication.

Definition 1.2 A semihomogeneous Banach space T is called convolutable if, for $g \in L^1(S,\pi)$, and $f \in T$, for the ordinary convolution product in $L^1(S,\pi)$ holds, that $f * g \in T$ and $\|f * g\|_T \leq \|f\|_T \|g\|_{L^1}$.

Hence, convolution w.r.t. $L^1(S,\pi)$ induces a module operation of $L^1(S,\pi)$ on convolutable T.

The Fourier-Jacobi transformation from $L^1(S,\pi)$ into $c_0(\mathbb{N}_0)$ is given by

$$\mathcal{F}(f)(n) = \int_{-1}^{1} f(z) R_n(z)\, d\pi(z). \tag{1.4}$$

We frequently use the notation $\check{f}$ instead of $\mathcal{F}(f)$. A uniqueness theorem is valid, and the range space is well defined, since a Riemann-Lebesgue lemma holds true. It can be easily shown that $\mathcal{F}(T_x f)(n) = R_n(x)\check{f}(n)$, for $n \in \mathbb{N}_0$ and $f \in L^1(S,\pi)$. The connection between convolution and Fourier-Jacobi transformation is given by

$$\mathcal{F}(f * g)(n) = \check{f}(n)\, \check{g}(n), \tag{1.5}$$

where $f, g \in L^1(S,\pi)$ and $n \in \mathbb{N}_0$.

We need the notion of an approximation kernel in $L^1(S,\pi)$ later on. Therefore we recall the definition.

Definition 1.3 Let I equal either $\mathbb{N}$ or $\mathbb{R}^+$. A family $\{K_i\}_{i\in I} \subset L^1(S,\pi)$ is called an approximation kernel if

(K1) $\int_{-1}^{1} K_i(x)\, d\pi(x) = 1$ for all $i \in I$.
(K2) $\sup_{i\in I}\ \|K_i\|_{L^1(S,\pi)} < \infty$,
(K3) $\lim_{i\to\infty} \check{K}_i(m) = 1$ for all $m \in \mathbb{N}_0$.

An approximation kernel is called positive if, in addition,

(K4) $K_i \geq 0$ for all $i \in I$.

We call the family $\{K_i\}_{i\in I}$ a polynomial approximation kernel, if K_i is a polynomial for every $i \in I$. An approximation kernel has the peaking property, if for every $h \in [-1,1[$, and every $\varepsilon > 0$, there is an $N \in I$, such that $\int_{-1}^{h} |K_n(x)|\, d\pi(x) < \varepsilon$ for all $n \geq N$.

The de la Vallée-Poussin kernel is an example of a positive polynomial approximation kernel, indexed by the nonnegative integers, which has the peaking property, see [1].

The structure of our paper is as follows. We start out by investigating semihomogeneous Banach spaces. We show that there is exactly one homogeneous Banach space contained in semihomogeneous T, which coincides with the closure of the space of polynomials w.r.t. the norm of T. Then we study an unbounded operator, the Jacobi differential operator, in a homogeneous Banach space B. This operator will be needed later on, and its domain is an example of a homogeneous Banach space. In the following we concentrate on Lipschitz spaces. Again starting from a homogeneous Banach space B, we can define little and big Lipschitz space w.r.t. B. The big Lipschitz spaces turn out to be convolutable semihomogeneous Banach spaces and the little Lipschitz spaces are the homogeneous ones contained in them. We characterize Lipschitz spaces in terms of intermediate spaces between a homogeneous Banach space B and the domain of the Jacobi differential operator in B. The rest of the paper is devoted to the proof that the second dual of a little Lipschitz space is the according big one, for general B. For $[0, 1]$ and the torus, with ordinary translation, these are results of Ciesielsky [8] and deLeeuw [15].

2 Semihomogeneous Banach Spaces

In this section we show that a homogeneous Banach space B is convolutable and that approximation with kernels also works in B. We also show that every semihomogeneous Banach space has exactly one closed homogeneous Banach subspace.

First of all, we show that the convolution of an element $g \in L^1(S, \pi)$ and $f \in B$, where B is a homogeneous Banach space, agrees with a module operation of $L^1(S, \pi)$ on B, defined by a certain Bochner integral.

Lemma 2.1 *Let B be a homogeneous Banach space. Then B is a Banach-$L^1(S, \pi)$-module via $\circledast : B \times L^1(S, \pi) \to B$, given by*

$$f \circledast g := \int_{-1}^{1} g(x) T_x f \, d\pi(x),$$

where the integral is understood as a Bochner integral. For all $f \in B$ and $g \in L^1(S, \pi)$, we have $f \circledast g = f * g$, where $*$ is the convolution in $L^1(S, \pi)$. In particular,

$$\|f * g\|_B \leq \|f\|_{L^1(S,\pi)} \|g\|_B. \tag{2.1}$$

P r o o f. Property (B4) of a homogeneous Banach space guarantees that $\{T_x f\}_{x \in S} \subset B$ is separable. Hence the function $[-1, 1] \to B,\ x \mapsto g(x) T_x f$, is strongly measurable. It follows from

$$\int_{-1}^{1} |g(x)| \, \|T_x f\|_B \, d\pi(x) \leq \int_{-1}^{1} |g(x)| \, \|f\|_B \, d\pi(x) = \|g\|_1 \, \|f\|_B$$

that it is Bochner integrable. To see that $\circledast$ coincides with the convolution, note that $B \to \mathbb{C},\ f \mapsto \check{f}(n)$, is continuous, since $|\check{f}(n)| \leq \|f\|_{L^1(S,\pi)} \leq \|f\|_B$. Hence, for $n \in \mathbb{N}_0$,

$$\begin{aligned}
\mathcal{F}(f \circledast g)(n) &= \int_{-1}^{1} g(x) \mathcal{F}(T_x f)(n) \, d\pi(x) \\
&= \int_{-1}^{1} g(x) R_n(x) \check{f}(n) \, d\pi(x) = \check{f}(n) \check{g}(n) = \mathcal{F}(f * g)(n).
\end{aligned}$$

The uniqueness theorem yields $f * g = f \circledast g \in B$. □

Proposition 2.2 *Let B be a homogeneous Banach space. Then the polynomials are dense in B.*

P r o o f. Let $f \in B$ and consider a positive polynomial approximation kernel $\{K_n\}_{n \in \mathbb{N}}$ with the peaking property. Since $K_n * f$ is a polynomial, it is sufficient to show that $\|K_n * f - f\|_B \to 0$ as $n \to \infty$. It follows from Lemma 2.1 that

$$\|K_n * f - f\|_B = \| \textstyle\int_{-1}^{1} (T_x f - f) K_n(x) \, d\pi(x) \|_B \leq \textstyle\int_{-1}^{1} \|T_x f - f\|_B K_n(x) \, d\pi(x).$$

Let $\varepsilon > 0$. By (B4), there is $h \in [-1, 1[$ such that $\|T_x f - f\|_B < \frac{\varepsilon}{4}$, for all x with $h \leq x < 1$. We obtain that

$$\begin{aligned}\|K_n * f - f\|_B &\leq \int_{-1}^{h} K_n(x)\|T_x f - f\|_B \, d\pi(x) + \int_{h}^{1} \frac{\varepsilon}{4} \, K_n(x) \, d\pi(x) \\ &\leq \int_{-1}^{h} (\|T_x f\|_B + \|f\|_B) \, K_n(x) \, d\pi(x) + \frac{\varepsilon}{2}.\end{aligned}$$

We use the peaking property of the kernel and choose $N \in \mathbb{N}$ such that $\int_{-1}^{h} K_m(x) \, d\pi(x) < \frac{\varepsilon}{4\|f\|_B}$, for all $m \geq N$. This implies

$$\|K_m * f - f\|_B \leq 2\|f\|_B \int_{-1}^{h} K_m(x) \, d\pi(x) + \frac{\varepsilon}{2} < 2\|f\|_B \, \frac{\varepsilon}{4\|f\|_B} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. □

An application of the Banach-Steinhaus theorem yields the following corollary.

Corollary 2.3 *Let B be a homogeneous Banach space and $\{K_n\}_{n \in I}$ be an approximation kernel. Seen as operators $K_n : B \to B$, $K_n f := K_n * f$, $n \in I$, we have that $\|K_n\|_{B \to B} \leq \|K_n\|_{L^1(S,\pi)}$. In addition, for all $f \in B$,*

$$\|K_n * f - f\|_B \to 0 \qquad \textit{as} \qquad n \to \infty.$$

We want to show that a semihomogeneous Banach space contains exactly one closed subspace which is a homogeneous Banach space. For a semihomogeneous Banach space T, we define B_T by

$$B_T = \{f \in T : \ x \mapsto T_x f \text{ is continuous in all } x \in [-1, 1]\} \, . \tag{2.2}$$

Lemma 2.4 *Let T be a semihomogeneous Banach space. Then B_T is a homogeneous Banach space.*

P r o o f. B_T is obviously a linear space. We start by showing that B_T is closed in T : Let $\{f_n\}_n$ be a sequence in B_T with $f_n \to f$ and let $\varepsilon > 0$. Furthermore, let $x_0 \in [-1, 1]$, and choose $N \in \mathbb{N}$ such that $\|f_N - f\|_T < \frac{\varepsilon}{3}$. Then $\|T_y f_N - T_y f\|_T \leq \|f_N - f\|_T < \frac{\varepsilon}{3}$ for all $y \in [-1, 1]$. Fix x_1 such that $\|T_x f_N - T_{x_0} f_N\|_T < \frac{\varepsilon}{3}$ for all $x \in \tilde{B}(x_0, |x_1 - x_0|) := \{x \in [-1, 1] : \ |x - x_0| \leq |x_1 - x_0|\}$. It follows that

$$\begin{aligned}&\|T_x f - T_{x_0} f\|_T \leq \|T_x f - T_x f_N\|_T + \|T_x f_N - T_{x_0} f_N\|_T + \|T_{x_0} f_N - T_{x_0} f\|_T \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \qquad \text{for all } \ x \in \tilde{B}(x_0, |x_1 - x_0|).\end{aligned}$$

For a Jacobi polynomial R_n $(n \in \mathbb{N}_0)$, and for $x, x_0 \in [-1, 1]$ we have that $T_x R_n - T_{x_0} R_n = (R_n(x) - R_n(x_0)) R_n$. Since polynomials are continuous and scalar multiplication is a continuous operation, $R_n \in B_T$, for all $n \in \mathbb{N}_0$. Property (B4) is clear by the definition of B_T. So it remains to show (B3), i.e., $x \mapsto T_x T_y f$ is continuous for $f \in B_T$ and for all $y \in [-1, 1]$. But this is true since

$$\|T_x T_y f - T_{x_0} T_y f\| = \|T_y T_x f - T_y T_{x_0} f\| \leq \|T_x f - T_{x_0} f\| \overset{x \to x_0}{\to} 0.$$

This completes the proof. □

Lemma 2.5 *Let T be a semihomogeneous Banach space. Then $\overline{span\,\{R_k : \ k \in \mathbb{N}_0\}} = B_T$.*

P r o o f. As $\{R_k : \ k \in \mathbb{N}_0\} \subset B_T$ and B_T is closed, it is sufficient to show that $B_T \subset \overline{span\,\{R_k : \ k \in \mathbb{N}_0\}}$. This is true since B_T is a homogeneous Banach space, which implies that the R_k are total by Corollary 2.3. □

Corollary 2.6 *In a semihomogeneous Banach space T, B_T is the only closed subspace which is a homogeneous Banach space.*

We call B_T the homogeneous Banach space in T.

Proposition 2.7 *Let T be a convolutable semihomogeneous Banach space. Then $T * L^1(S,\pi) = B_T$.*

Proof. We let $B = T * L^1(S,\pi)$. Since T is convolutable we can regard T as an Banach-$L^1(S,\pi)$-module. Now Cohen's module factorization theorem implies that B is a closed subspace of T. Since $R_n = h_n R_n * R_n \in T * L^1(S,\pi)$ the polynomials are in B. For $f \in B$ and $g \in L^1(S,\pi)$ we use $T_x(f*g) = f * T_x g \in T * L^1(S,\pi) = B$ to show that

$$\|T_x(f*g) - T_{x_0}(f*g)\|_T = \|f * T_x g - f * T_{x_0} g\|_T \leq \|T_x g - T_{x_0} g\|_{L^1} \|f\|_T.$$

The right hand expression approaches 0, as $x \to x_0$ in S, since $L^1(S,\pi)$ is a homogeneous Banach space. Therefore $B_T = \overline{span\,\{R_k : \ k \in \mathbb{N}_0\}} \subset B \subset B_T$. □

3 Jacobi Differential Operator

We introduce a closed, densely defined unbounded operator in a homogeneous Banach space B here. We call that operator the Jacobi differential operator in B. For the spaces $L^p(S,\pi)$, $1 \leq p < \infty$, and $C(S)$, this operator has been considered by Bavinck [1] and Pawelke [17]. In [10], this operator is treated on $L^2(S,\pi)$. We need this operator in Section 4.2, where we characterize Lipschitz spaces as intermediate spaces w.r.t. B and the domain of this operator.

Definition 3.1 Let B be a homogeneous Banach space. We let

$$D_A = \left\{ f \in B : \ \exists\, g \in B \text{ such that } \check{g}(n) = -n(n+\alpha+\beta+1)\check{f}(n) \ \text{ for all } n \in \mathbb{N}_0 \right\}$$

and define the Jacobi Differential Operator $A : \ D_A \to B,\ f \mapsto g$.

Proposition 3.2 *Let $B,\ D_A$ and A be as in Definition 3.1. Then $A : \ D_A \to B$ is a closed and densely defined linear unbounded operator in B. Its domain D_A, with the graph norm $\|f\|_{D_A} := \|f\|_B + \|Af\|_B$, is a homogeneous Banach space.*

Proof. The linearity of A is obvious. For every Jacobi polynomial R_n $(n \in \mathbb{N}_0)$, we have that $-n(n+\alpha+\beta+1)R_n \in B$. Hence

$$R_n \in D_A \qquad \text{and} \qquad AR_n = -n(n+\alpha+\beta+1)R_n, \tag{3.1}$$

and we see that A is unbounded. Since the polynomials are dense in B, A is densely defined. In order to show that A is a closed operator, we let $(f_n, Af_n)_n$ be a sequence in the graph G_A of A with $(f_n, Af_n) \to (f,g)$ in $B \times B$. As the mapping $B \to \mathbb{C},\ h \mapsto \check{h}(k)$, is continuous we have $\check{f}_n(k) \to \check{f}(k)$ and $-k(k+\alpha+\beta+1)\check{f}_n(k) \to \check{g}(k)$. Thus,

$$\check{g}(k) = \lim_{n\to\infty} -k(k+\alpha+\beta+1)\check{f}_n(k) = -k(k+\alpha+\beta+1)\check{f}(k).$$

It follows that $f \in D_A$ and $Af = g$.

We now show that D_A is a homogeneous Banach space. From the definition of the graph norm and the fact that A is a closed operator, we can deduce that $(D_A, \|\ \|_{D_A})$ is a Banach space contained in $L^1(S,\pi)$ and that for $f \in D_A$ we have $\|f\|_{D_A} \geq \|f\|_B \geq \|f\|_{L^1(S,\pi)}$. So (B2) is fulfilled, and (B1) follows from (3.1). We show (B3): Suppose that $f \in D_A$. Then $Af \in B$, and since B is a homogeneous Banach space we can show that $T_x Af \in B$, for all $x \in S$, as follows: By Fourier expansion, we obtain that, for all $n \in \mathbb{N}_0$,

$$\mathcal{F}(T_x Af)(n) = -n(n+\alpha+\beta+1)R_n(x)\check{f}(n) = -n(n+\alpha+\beta+1)\mathcal{F}(T_x f)(n).$$

This implies $T_x f \in D_A$, and $AT_x f = T_x Af$, for all $x \in S$. Since B is a homogeneous Banach space, we get

$$\|T_x f\|_{D_A} = \|T_x f\|_B + \|AT_x f\|_B \leq \|f\|_B + \|T_x Af\|_B \leq \|f\|_{D_A}.$$

Furthermore, for $x_0 \in S$,

$$\|T_x f - T_{x_0} f\|_{D_A} = \|T_x f - T_{x_0} f\|_B + \|T_x Af - T_{x_0} Af\|_B \to 0$$

as $x \to x_0$, which yields (B4). □

It is well known that the Jacobi polynomials fulfill the equality

$$\frac{d}{dx}\left(w(x)(1-x^2)\frac{d}{dx}R_n^{(\alpha,\beta)}(x)\right) = -n(n+\alpha+\beta+1)w(x)R_n^{(\alpha,\beta)}(x). \tag{3.2}$$

The following goes back to Löfström and Peetre, see [16]. It also works for general homogeneous Banach spaces. Integrating (3.2) with bounds in the interior of the interval and letting the upper bound to 1 yields for $x \in S$

$$R_n(1) - R_n(x) = n(n+\alpha+\beta+1)\int_x^1 \tfrac{1}{w(t)(1-t^2)} \int_t^1 R_n(s)w(s)\, dsdt. \tag{3.3}$$

We consider the function $\theta : [0,1]\times[0,1] \to [0,\infty[$,

$$\theta(x,s) = \begin{cases} \frac{1}{c_{\alpha,\beta}}\int_x^s \frac{1}{w(t)(1-t^2)}\, dt & \text{for } 1 > s > x, \\ 0 & \text{otherwise,} \end{cases}$$

and the function $C : [0,1] \to [0,\infty[$, $C(x) = \int_x^1 \theta(x,s)\, d\pi(s)$. With the help of Tonnelli's theorem it follows that $s \mapsto \theta(x,s)$ is π-integrable, and a direct calculation yields the estimate:

$$m(1-x) \le C(x) \le M(1-x), \qquad \text{for all } x \in [0,1], \tag{3.4}$$

where m and M are positive constants.

Lemma 3.3 *Let B be a homogeneous Banach space. Then, for all $x \in [0,1[$, the operator $I_x : B \to B$,*

$$I_x f := \frac{1}{C(x)}\int_x^1 \theta(x,s)T_s f\, d\pi(s), \tag{3.5}$$

is a linear contraction on B.

Proof. Since B is a homogeneous Banach space and $s \mapsto \theta(x,s)$ is π-integrable, the integrand is strongly measurable. Furthermore,

$$\|I_x f\|_B \le \frac{1}{C(x)}\int_x^1 \theta(x,s)\|T_s f\|_B\, d\pi(s) \le \|f\|_B.$$

Thus $I_x f \in B$ and I_x is a contraction. I_x is linear, since so are translation and integral. □

Lemma 3.4 *Let B be a homogeneous Banach space and A be the Jacobi differential operator in B. Let furthermore $\{I_x\}_{0\le x<1}$ be as defined in Lemma 3.3. Then the following statements are true for any $x \in [0,1[$,*

1. *For any $h \in D_A,\ I_x A h = \frac{T_x h - h}{C(x)}$.*
2. *For any $f \in B,\ I_x f \in D_A$ and $T_x f - f = C(x) A I_x f$.*
3. *The linear operator $A \circ I_x : B \to B$ is bounded, and $A \circ I_x = I_x \circ A$ on D_A.*

Proof. Beginning with 2., we show that $I_x f \in D_A$. We calculate the Fourier coefficients of $C(x)I_x f$. For all $n \in \mathbb{N}_0$, we get

$$\begin{aligned} &-n(n+\alpha+\beta+1)\mathcal{F}(C(x)I_x f)(n) = -n(n+\alpha+\beta+1)\int_x^1 \theta(x,s)\mathcal{F}(T_s f)(n)\, d\pi(s) \\ &= -n(n+\alpha+\beta+1)\int_x^1 \theta(x,s)R_n(s)\, d\pi(s)\check{f}(n) \\ &= -\left[R_n(1) - R_n(x)\right]\check{f}(n) = \mathcal{F}(T_x f - f)(n), \end{aligned}$$

where we used (3.3) for the last but one equality. Since $T_x f - f \in B$ we have $C(x)I_x f \in D_A$. The uniqueness theorem guarantees that $AC(x)I_x f = T_x f - f$. The first statement of 3. is a consequence of

$$\|AI_x f\|_B = \frac{1}{C(x)}\|T_x f - f\|_B \le \frac{2}{C(x)}\|f\|_B.$$

An analogous calculation of the Fourier coefficients of $Ah \in B$ instead of f yields 1. For $h \in D_A$, we combine 1. and 2. and obtain that $C(x)I_x Ah = T_x h - h = C(x)AI_x h$. This proves the remainder of 3. □

4 Lipschitz Spaces

Lipschitz spaces w.r.t. Jacobi translation were defined in [10] in the following way. Let B be a homogeneous Banach space and assume that $0 < \lambda < 1$. We define

$$\mathrm{Lip}_B(\lambda) = \left\{ f \in B : \sup_{x \in [-1,1[} \frac{\|T_x f - f\|_B}{(1-x)^\lambda} < \infty \right\}.$$

$\mathrm{Lip}_B(\lambda)$ becomes a Banach space with the norm $\|f\|_{\mathrm{Lip}} = \|f\|_B + \sup_{x\in[-1,1[} \frac{\|T_x f - f\|_B}{(1-x)^\lambda}$. For the special choices $B = L^p(S,\pi)$, with $1 \le p < \infty$, and $B = C(S)$ these spaces are certain Jacobi-Besov spaces. We exemplarily refer to Bavinck [1], Runst and Sickel [18], and Kyriazis et al. [13] for further work on function spaces related to Jacobi polynomials.

The space $\mathrm{lip}_B(\lambda)$ is the subspace of $\mathrm{Lip}_B(\lambda)$ given by

$$\mathrm{lip}_B(\lambda) = \left\{ f \in B : \lim_{x\to 1} \tfrac{\|T_x f - f\|_B}{(1-x)^\lambda} = 0 \right\}.$$

Proposition 4.1 *Let B be a homogeneous Banach space and $0 < \lambda < 1$. Then $\mathrm{Lip}_B(\lambda)$ is a convolutable semihomogeneous Banach space and the closed subspace $\mathrm{lip}_B(\lambda)$ is a homogeneous Banach space.*

Proof. In [10], it is shown that $\mathrm{lip}_B(\lambda)$ is a homogeneous Banach space. It is implicitly proved there that $\mathrm{Lip}_B(\lambda)$ is a Banach space which is contained in $L^1(S,\pi)$, and that $\mathrm{lip}_B(\lambda)$ is a closed subspace. It also follows from [10] that $R_n \in \mathrm{lip}_B(\lambda) \subset \mathrm{Lip}_B(\lambda)$, and that, for $f \in \mathrm{Lip}_B(\lambda)$, and $x, y \in S$, we have $\|T_y(T_x f) - T_x f\|_B \le \|T_y f - f\|_B$. This implies $T_x f \in \mathrm{Lip}_B(\lambda)$ with $\|T_x f\|_{\mathrm{Lip}} \le \|f\|_{\mathrm{Lip}}$.

To see that $\mathrm{Lip}_B(\lambda)$ is convolutable, let $f \in \mathrm{Lip}_B(\lambda)$ and $g \in L^1(S,\pi)$. Then,

$$\|T_x(f*g) - f*g\|_B = \|(T_x f - f)*g\|_B \le \|T_x f - f.\|_B \|g\|_{L^1}$$

This implies $f*g \in \mathrm{Lip}_B(\lambda)$, and $\|f*g\|_{\mathrm{Lip}} \le \|f\|_{\mathrm{Lip}}\|g\|_{L^1}$. □

From Lemma 2.5, Corollary 2.6 and Proposition 2.7 we can deduce the following relation between the little and big Lipschitz spaces.

Corollary 4.2 *Let B be a homogeneous Banach space and $0 < \lambda < 1$. Then $\mathrm{lip}_B(\lambda)$ is the homogeneous Banach space in $\mathrm{Lip}_B(\lambda)$, in particular $\mathrm{lip}_B(\lambda)$ is the closure of the polynomials in $\mathrm{Lip}_B(\lambda)$. Furthermore,*

$$\mathrm{Lip}_B(\lambda) * L^1(S,\pi) = \mathrm{lip}_B(\lambda). \tag{4.1}$$

4.1 Equivalent norms based on a modulus of continuity

For a homogeneous Banach space B and $t \in [-1,1[$, a modulus of continuity of $f \in B$ is given by

$$w_B(f,t) = \sup_{t \le x \le 1} \|T_x f - f\|_B. \tag{4.2}$$

Obviously, $w_B(f,t)$ is a finite non-negative number. For fixed $f \in B$, $t \mapsto w_B(f,t)$, is non-increasing in S, and $w_B(f,t) \to 0$ for $t \uparrow 1$. It is also easily seen that $w_B(f+g,t) \le w_B(f,t) + w_B(g,t)$. This justifies calling w_B modulus of continuity. Note that this modulus of continuity does not fit into the framework of the book of Ditzian and Totik [9], which is due to the Jacobi translation employed in its definition.

Definition 4.3 We say that a sequence $d = \{d_n\}_{n\in\mathbb{N}_0}$ of real numbers is of type $[E]$, if $d_0 = -1$, d is strictly increasing, $\lim_{n\to\infty} d_n = 1$, and there is $C_d > 0$ such that $\frac{1-d_n}{1-d_{n+1}} \le C_d$ for all $n \in \mathbb{N}_0$.

We define, for $\varepsilon > 0$, the sequence d by $d_n := 1 - 2(\frac{1}{1+\varepsilon})^n$ $(n \in \mathbb{N}_0)$. Then d is a sequence of type $[E]$ with

$$C_d = \sup_{n\in\mathbb{N}_0} \tfrac{1-d_n}{1-d_{n+1}} \le 1 + \varepsilon. \tag{4.3}$$

Hence we can choose C_d arbitrarily close to 1.

Lemma 4.4 *Let B be a homogeneous Banach space, $\lambda \in]0,1[$, and $d = \{d_n\}_{n\in\mathbb{N}_0}$ be a sequence of type $[E]$. Then,*

$$\|f\|'_{\mathrm{Lip}} = \|f\|_B + \sup_{t\in[-1,1]} \tfrac{w_B(f,t)}{(1-t)^\lambda}, \qquad \textit{and} \qquad \|f\|_{\mathrm{Lip},d} = \|f\|_B + \sup_{n\in\mathbb{N}_0} \tfrac{w_B(f,d_n)}{(1-d_n)^\lambda}$$

define equivalent norms on $\mathrm{Lip}_B(\lambda)$. We have the equality $\|f\|'_{\mathrm{Lip}} = \|f\|_{\mathrm{Lip}}$ and $\frac{1}{C_d}\|f\|_{\mathrm{Lip}} \le \|f\|_{\mathrm{Lip},d} \le \|f\|_{\mathrm{Lip}}$, for all $f \in \mathrm{Lip}_B(\lambda)$. The space $\mathrm{lip}_B(\lambda)$ is characterized by

$$\mathrm{lip}_B(\lambda) = \left\{ f \in B:\ \lim_{t\uparrow 1} \tfrac{w_B(f,t)}{(1-t)^\lambda} = 0 \right\} = \left\{ f \in B:\ \lim_{n\to\infty} \tfrac{w_B(f,d_n)}{(1-d_n)^\lambda} = 0 \right\}.$$

P r o o f. It is clear that $\|\cdot\|'_{\mathrm{Lip}}$ and $\|\cdot\|_{\mathrm{Lip},d}$ are norms on $\mathrm{Lip}_B(\lambda)$, and that $\|\cdot\|_{\mathrm{Lip}} \le \|\cdot\|'_{\mathrm{Lip}}$ and $\|\cdot\|_{\mathrm{Lip},d} \le \|\cdot\|'_{\mathrm{Lip}}$. For $t \in [-1,1[$ and $f \in \mathrm{Lip}_B(\lambda)$, we have $\frac{w_B(f,t)}{(1-t)^\lambda} = \sup_{t\le x<1} \frac{\|T_x f - f\|}{(1-t)^\lambda} \le \sup_{t\le x<1} \frac{\|T_x f-f\|}{(1-x)^\lambda}$. It follows that $\|f\|'_{\mathrm{Lip}} \le \|f\|_{\mathrm{Lip}}$. Finally, choose $t \in [-1,1[$. Then there is $N \in \mathbb{N}$ such that $d_N < t \le d_{N+1}$, and we have

$$\tfrac{w_B(f,t)}{(1-t)^\lambda} \le \tfrac{w_B(f,d_N)}{(1-d_{N+1})^\lambda} \le \tfrac{1-d_N}{1-d_{N+1}} \tfrac{w_B(f,d_N)}{(1-d_N)^\lambda} \le \sup_{n\in\mathbb{N}_0} \tfrac{1-d_n}{1-d_{n+1}} \tfrac{w_B(f,d_n)}{(1-d_n)^\lambda} \le C_d \|f\|_{\mathrm{Lip},d},$$

where we used the monotonicity of the modulus of continuity. Therefore, $\|f\|_{\mathrm{Lip}} \le C_d \|f\|_{\mathrm{Lip},d}$. The statement on the little Lipschitz spaces is clear. □

4.2 Lipschitz spaces as Intermediate Spaces

The purpose of the following is to characterize Lipschitz spaces in terms of K-intermediate spaces. References concerning interpolation are the books of Bergh and Löfström [3] and of Triebel [20].

We begin by introducing Peetre's K-functional. We consider a Banach space X and let $Y \subset X$ be a normalized Banach subspace of X, i. e. a linear subspace which becomes a Banach space with its norm $\|\cdot\|_Y$ such that $\|f\|_X \le \|f\|_Y$ for all $f \in Y$. Then the K-functional $K_{X,Y}:\ X\times]0,1] \to [0,\infty[$ is given by

$$K_{X,Y}(f,t) := \inf_{g\in Y} \|f-g\|_X + t\|g\|_Y. \tag{4.4}$$

We write K instead of $K_{X,Y}$ when there is no danger of confusion. Information on the K-functional and the following simple properties can be found e.g. in [6]. For fixed t, $K(\cdot,t)$ is a norm equivalent to $\|\cdot\|_X$ with bounds

$$t\|f\|_X \le K(f,t) \le \|f\|_X \qquad \text{for all } f \in X. \tag{4.5}$$

Furthermore, for fixed $f \in X$, the mapping $t \mapsto K(f,t)$, is non-decreasing and concave. For $h \in Y$, we have

$$K(h,t) \le t\|h\|_Y. \tag{4.6}$$

We define, for $\lambda \in]0,1[$,

$$(X,Y)_{\lambda,K} = \left\{ f \in X:\ \sup_{n\in\mathbb{N}} n^\lambda K(f,\tfrac{1}{n}) < \infty \right\}. \tag{4.7}$$

In [7], Butzer and Scherer denote this space by $[X,Y]^+_{\theta,\infty,K}$. They show that $(X,Y)_{\lambda,K}$ is a Banach space with the norm $\|f\|_{\lambda,K} = \sup_{n\in\mathbb{N}} n^\lambda K(f,\frac{1}{n})$. Furthermore, $Y \subset (X,Y)_{\lambda,K} \subset X$ with continuous embeddings. In addition, $(X,Y)_{\lambda,K}$ is contained in the closure of Y with respect to $\|\cdot\|_X$.

In our setting, the part of the space X above is played by a homogeneous Banach space B, whereas the role of Y will be taken by the Jacobi differential operator from Section 3. The following proposition is known for $L^p(S,\pi)$, $1 \le p < \infty$, and $C(S)$, see [1] and [17].

Proposition 4.5 *Let B be a homogeneous Banach space, $A:\ D_A \to B$ be the Jacobi differential operator in B, and D_A be equipped with its graph norm. Then there are constants $m, M > 0$ such that, for all $f \in B$, and all $t \in [-1,1[$,*

$$m\left(w(f,t) + \tfrac{1-t}{2}\|f\|_B\right) \le K(f,\tfrac{1-t}{2}) \le M\left(w(f,t) + \tfrac{1-t}{2}\|f\|_B\right). \tag{4.8}$$

P r o o f. At first, consider $t \in [-1, 0]$. We can estimate the K-functional from both above and below by a positive constant times $\|\cdot\|_B$. We can also estimate the modulus of continuity from above by $\|\cdot\|_B$ multiplied with a positive constant. This implies that (4.8) is valid for $t \in [-1, 0]$. So we can restrict to $t \in [0, 1[$ in the following.

For the remainder of this proof, let m_i, resp. M_i, always denote positive constants which are independent of $f \in B$ and $t \in [0, 1[$. We begin with the first inequality. It follows from (4.5) that $\frac{1-t}{2}\|f\|_B \le K(f, \frac{1-t}{2})$. Thus it is enough to find $m_1 > 0$ such that, for all $f \in B$,

$$w(f, t) \le m_1\, K(f, \tfrac{1-t}{2}). \tag{4.9}$$

For $g \in D_A$ and $x \in [0, 1[$, Lemma 3.4.1 implies that $T_x g - g = C(x) I_x A g$. I_x is a contraction, and we use (3.4) to find m_2, such that

$$\|T_x g - g\|_B \le C(x)\|Ag\|_B \le m_2(1-x)\|Ag\|_B.$$

Therefore there is $m_3 > 0$ such that $w(g, t) = \sup_{t\le x\le 1} \|T_x g - g\|_B \le m_3 \frac{1-t}{2}\ \|Ag\|_B$. So, for $f \in B$, and $g \in D_A$, we have, by the properties of the modulus of continuity, that

$$\begin{aligned} w(f, t) &\le w(f-g, t) + w(g, t) \le 2\|f-g\|_B + m_3 \tfrac{1-t}{2}\ \|Ag\|_B \\ &\le \max(2, m_3)\left(\|f-g\|_B + \tfrac{1-t}{2}\ \|g\|_{D_A}\right). \end{aligned}$$

Passing to the infimum with respect to $g \in D_A$ yields (4.9).

In order to show the second inequality in (4.8), choose $f \in B$. Consider, for $t \in [0, 1[$, the element $I_t f$, where I_t is given in Lemma 3.3. From Lemma 3.4.2 we deduce that $I_t f \in D_A$, and that

$$I_t f - f = \tfrac{1}{C(t)} \int_t^1 \theta(t, s)(T_s f - f)\, d\pi(s).$$

It follows that

$$\|I_t f - f\|_B \le \tfrac{1}{C(t)} \int_t^1 \theta(t, s)\|T_s f - f\|_B\, d\pi(s) \le \sup_{t\le s<1}\ \|T_s f - f\|_B = w(f, t).$$

We use again (3.4) and find M_1 such that

$$\begin{aligned} K\left(f, \tfrac{1-t}{2}\right) &\le \|I_t f - f\|_B + \tfrac{1-t}{2}\|I_t f\|_{D_A} \le w(f, t) + \tfrac{1-t}{2}\left(\left\|\tfrac{1}{C(t)}(T_t f - f)\right\|_B + \|I_t f\|_B\right) \\ &\le w(f, t) + M_1 w(f, t) + \tfrac{1-t}{2}\|f\|_B, \end{aligned}$$

which completes the proof. □

Corollary 4.6 *Let B be a homogeneous Banach space, $\lambda \in]0, 1[$, $A : D_A \to B$ be the Jacobi differential operator in B and D_A be equipped with its graph norm. Then*

$$(B, D_A)_{\lambda,K} \cong \mathrm{Lip}_B(\lambda), \tag{4.10}$$

i. e. they are equal as sets and their norms are equivalent. Furthermore, $\mathrm{lip}_B(\lambda)$ corresponds to the closed subspace

$$(B, D_A)^0_{\lambda,K} := \left\{f \in (X; D_A)_{\lambda,K} :\ \lim_{n\to\infty} n^\lambda K(f, \tfrac{1}{n}) = 0\right\}. \tag{4.11}$$

P r o o f. Define the sequence $d = \{d_n\}_{n\in\mathbb{N}_0}$ of real numbers by $d_n := 1 - \frac{2}{n+1}$. Then d is of type $[E]$ and Lemma 4.4 ensures that $\|\cdot\|_{\mathrm{Lip},d}$ is an equivalent norm on $\mathrm{Lip}_B(\lambda)$. Now a calculation using Proposition 4.5 yields (4.10). In order to see (4.11) take $g \in \mathrm{lip}_B(\lambda)$. We have, for all $n \in \mathbb{N}$,

$$n^\lambda K\left(g, \tfrac{1}{n}\right) \le M n^\lambda w\left(g, 1 - \tfrac{2}{n}\right) + M \tfrac{1}{n^{1-\lambda}}\|g\|_B,$$

where $M > 0$ is independent of g and n. On the one hand, we get that $g \in (B, D_A)^0_{\lambda,K}$, since both summands on the right-hand side approach zero for $n \to \infty$. On the other hand, if $g \in (B, D_A)^0_{\lambda,K}$, (4.8) implies that $n^\lambda w\left(g, 1 - \frac{2}{n}\right) \to 0$ for $n \to \infty$ and Lemma 4.4 tells us that $g \in \mathrm{lip}_B(\lambda)$. □

5 The Second Dual of $\mathrm{lip}_B(\alpha)$

We begin this chapter by showing that the embedding $i:\ (D_A, \|\cdot\|_{D_A}) \to (B, \|\cdot\|_B)$ is compact. Then we show that the embedding $j:\ \big((B, D_A)^0_{\alpha,K}, \|\cdot\|_{(B,D_A)_{\alpha,K}}\big) \to (B, \|\cdot\|_B)$ is weakly compact, which allows us to characterize the second dual of $\mathrm{lip}_B(\lambda)$.

Definition 5.1 Let B be a homogeneous Banach space. The Weierstraß semigroup is defined as the family of operators $\{W_t\}_{t\geq 0}$ given by

$$W_t:\ B \to B, \qquad W_0 f = f, \quad W_t f = w_t * f \ \text{ for } t > 0,$$

where $w_t = \sum_{n=1}^{\infty} e^{-n(n+\alpha+\beta+1)t} R_n h_n$.

Since $h_n = O(n^{2\alpha+1})$ for $n \to \infty$ and $\sup_{n\in\mathbb{N}_0} \|R_n\|_{L^1(S,\pi)} \leq \sup_{n\in\mathbb{N}_0} \|R_n\|_\infty = 1$, w_t exists as an element of $L^1(S,\pi)$ for all $t > 0$ as an absolutely convergent series.

Proposition 5.2 *Let B be a homogeneous Banach space. Then the Weierstraß semigroup $\{W_t\}_{t\geq 0}$ is a C_0-semigroup of contractions on B, whose infinitesimal generator equals A, the Jacobi differential operator.*

The $\{w_t\}_{t>0}$ are positive in the ultraspherical case which is a result of Bochner (see Theorem 3.4.3 of [5]). Gasper pointed out its validity for Jacobi polynomials in [11]. Now it is easy to see that the family $\{w_{1/i}\}_{i>0}$ defines a positive approximation kernel and by Corollary 2.3 $\{W_t\}_{t\geq 0}$ is a C_0-semigroup of contractions on B. It can be shown that the infinitesimal generator of $\{W_t\}_{t\geq 0}$ is A, as pointed out by Bavinck [1]. The argument is similar to the one used by Butzer and Behrens in [6], Theorem 1.5.4. When we denote the infinitesimal generator by C and take $g \in D_C$, then computing Fourier coefficients of Cg yields $g \in D_A$, and if we suppose $g \in D_A$, we approximate g in D_A by convolution with a polynomial approximation kernel. These polynomials are elements of D_C and since $Ap = Cp$ for every polynomial p, the closedness of C ensures that $g \in D_C$.

Recall that a C_0-semigroup $\{U_t\}_{t\geq 0}$ in a Banach space is called compact, if for each $t > 0$ U_t is a compact operator.

Proposition 5.3 *The Weierstraß semigroup $\{W_t\}_{t\geq 0}$ on a homogeneous Banach space B is compact.*

P r o o f. We fix $t > 0$ and show that W_t is compact as the operator-norm limit of operators with finite dimensional range. Therefore, for $n \in \mathbb{N}$, we let

$$w_t^{(n)} := \sum_{k=0}^{n} e^{-k(k+\alpha+\beta+1)t} R_k h_k.$$

We let $W_t^{(n)}$ operate on B via $W_t^{(n)}(f) := w_t^{(n)} * f$. Then $W_t^{(n)}(B) \subset \mathit{span}_{0\leq k\leq n}\,\{R_k\}$, and we have

$$\begin{aligned}\|W_t - W_t^{(n)}\|_{B\to B} &= \sup\nolimits_{\|g\|_B\leq 1} \|w_t * g - w_t^{(n)} * g\|_B \\ &\leq \sup\nolimits_{\|g\|_B\leq 1} \|w_t - w_t^{(n)}\|_1 \|g\|_B = \|w_t - w_t^{(n)}\|_1.\end{aligned}$$

But

$$\begin{aligned}\|w_t - w_t^{(n)}\|_1 &= \Big\| \sum\nolimits_{k=n+1}^{\infty} e^{-k(k+\alpha+\beta+1)t} R_k h_k \Big\|_1 \\ &\leq \sup\nolimits_{k\in\mathbb{N}} \|R_k\|_1 \cdot \Big(\sum\nolimits_{k=n+1}^{\infty} e^{-k(k+\alpha+\beta+1)t} h_k \Big),\end{aligned}$$

and since $h_k = O(k^{2\alpha+1})$, the infinite scalar sum converges to zero as $n \to \infty$. This completes the proof. □

Proposition 5.4 *Let B be a homogeneous Banach space and let $A:\ D_A \to B$ be the Jacobi differential operator. Then the embedding $i:\ (D_A, \|\cdot\|_{D_A}) \to (B, \|\cdot\|_B)$ is compact.*

P r o o f. We define $H:\ (D_A, \|\cdot\|_{D_A}) \to (B, \|\cdot\|_B)$, $Hx := (I - A)x$, which is continuous since $\|Hx\|_B = \|x - Ax\|_B \leq \|x\|_B + \|Ax\|_B = \|x\|_{D_A}$. Now consider

$$(I - A)^{-1}:\ (B, \|\cdot\|_B) \to (D_A, \|\cdot\|_B).$$

By Proposition 5.3 the Weierstraß semigroup is compact and by Proposition 5.2 A is its infinitesimal generator. From Pazy's theorem it follows that $(I - A)^{-1}$ is compact. Hence $i = (I - A)^{-1} \circ H$ is compact. □

Recall that Gantmacher's theorem states that a bounded linear operator $T:\ X \to Y$, where X and Y are Banach spaces, is weakly compact if and only if its second adjoint T^{**} obeys: $T^{**}(X^{**}) \subset Q(Y)$, Q denoting the canonical embedding.

The following essentially uses adaptions of interpolation techniques used by Beauzamy in [2], chapitre II, there attributed to W. J. Davis, T. Figiel, W. B. Johnson and A. Pelczynski.

Proposition 5.5 *Let B be a homogeneous Banach space and $0 < \lambda < 1$. If the embedding $i:\ (D_A, \|\cdot\|_{D_A}) \to (B, \|\cdot\|_B)$ is weakly compact, then*

$$j:\ (\mathrm{lip}_B(\lambda), \|\cdot\|_{\mathrm{lip}}) \to (B, \|\cdot\|_B),\ j(f) := f,$$

is weakly compact.

P r o o f. In this proof $K_X(r)$ denotes the closed ball with radius r in the Banach space X. Let $f \in \mathrm{lip}_B(\lambda)$, such that $\|f\|_{\mathrm{lip}} \leq 1$. By Corollary 4.6 there is a constant $C > 0$, independent of f, such that for every $t \in [-1, 1[$ there is $g_t \in D_A$ with

$$(\tfrac{1-t}{2})^{-\lambda}\|f - g_t\|_B + (\tfrac{1-t}{2})^{1-\lambda}\|g_t\|_{D_A} \leq C\|f\|_{\mathrm{lip}} \leq C.$$

Therefore for every $t \in [-1, 1[$ we have

$$f = f - g_t + g_t \in K_B(C(\tfrac{1-t}{2})^{\lambda}) + K_{D_A}(C(\tfrac{1-t}{2})^{\lambda-1}). \tag{5.1}$$

Now we consider the balls in (5.1) as subsets of B^{**} via the canonical embedding Q. By assumption, the closure of $K_{D_A}(C(\frac{1-t}{2})^{\lambda-1})$ in B is weakly compact, hence its image under Q is weak-$*$-compact in B^{**}. Since addition is weak-$*$-continuous and $K_{B^{**}}(C(\frac{1-t}{2})^{\lambda})$ is weak-$*$-compact, we get that $K_{B^{**}}(C\frac{1-t}{2})^{\lambda} + Q(\overline{K_{D_A}(C(\frac{1-t}{2})^{\lambda-1})})$ is weak-$*$-compact, in particular weak-$*$-closed. Hence it follows that

$$\begin{aligned}\overline{Q(B_{\mathrm{lip}}(1))}^{w^*} &\subset \bigcap\nolimits_{t\in[-1,1[} K_{B^{**}}(C(\tfrac{1-t}{2})^{\lambda}) + Q\big(\overline{K_{D_A}(C(\tfrac{1-t}{2})^{\lambda-1})}\big)\\ &\subset \bigcap\nolimits_{t\in[-1,1[} K_{B^{**}}(C(\tfrac{1-t}{2})^{\lambda}) + Q(B) \quad = \quad Q(B).\end{aligned}$$

The weak-$*$-continuity of j^{**} and Goldstine's theorem yield

$$j^{**}(B_{\mathrm{lip}^{**}}(1)) = j^{**}(\overline{Q_{\mathrm{lip}}(B_{\mathrm{lip}}(1))}^{w^*}) \subset \overline{Q(j(B_{\mathrm{lip}}(1)))}^{w^*} \subset Q(B).$$

This completes the proof using Gantmacher's theorem. □

In the following we factor the embeddings of our Lipschitz spaces into the homogeneous Banach space through a sequence space of suitably chosen Banach spaces. For this purpose we consider a sequence of Banach spaces $\{X_n, \|\ \|_n\}_{n\in\mathbb{N}}$. We define

$$\Big(\sum\nolimits_{n\in\mathbb{N}} X_n\Big)_{\infty} := \{\{x_n\}_{n\in\mathbb{N}}:\ x_n \in X_n,\ \sup_{n\in\mathbb{N}}\|x_n\|_n < \infty\}.$$

$(\sum_{n\in\mathbb{N}} X_n)_\infty$ is a Banach space and

$$\Big(\sum\nolimits_{n\in\mathbb{N}} X_n\Big)_0 := \Big\{\{x_n\}_{n\in\mathbb{N}} \in \Big(\sum\nolimits_{n\in\mathbb{N}} X_n\Big)_{\infty}:\ \lim_{n\in\mathbb{N}}\|x_n\|_n = 0\Big\}$$

is a closed subspace. The p-sums $(\sum_{n\in\mathbb{N}} X_n)_p$ shall be defined analogously for $1 \leq p < \infty$. Similar to the scalar case, dualities can be implemented as follows:

$$\begin{aligned}&\Big(\sum\nolimits_n X_n\Big)_0^* \to \Big(\sum\nolimits_n X_n^*\Big)_1,\ x^* \mapsto \big\{x^*|_{(0,\dots,0,X_n,0,\dots)}\big\}_{n\in\mathbb{N}}, \text{ and}\\ &\Big(\sum\nolimits_n X_n^*\Big)_1^* \to \Big(\sum\nolimits_n X_n^{**}\Big)_{\infty},\ x^{**} \mapsto \big\{x^{**}|_{(0,\dots,0,X_n^*,0,\dots)}\big\}_{n\in\mathbb{N}}.\end{aligned}$$

Under these identifications the canonical embedding from $(\sum_{n\in\mathbb{N}} X_n)_0$ into its bidual reads

$$\left(\sum\nolimits_{n\in\mathbb{N}} X_n\right)_0 \to \left(\sum\nolimits_{n} X_n^{**}\right)_\infty,\ \{x_n\}_{n\in\mathbb{N}} \mapsto \{Qx_n\}_{n\in\mathbb{N}}.$$

We recall the following well known facts. If T is a topological isomorphism from a Banach space X into a Banach space Y, then so is $T^{**}:\ X^{**}\to Y^{**}$ which has image $T(X)^{\perp\perp} = \overline{Q(T(X))}^{w*}$. Furthermore, if there are bounds $C, c > 0$ such that $c\|f\| \le \|Tf\| \le C\|f\|$ for all $f \in B$, then also $c\|f^{**}\| \le \|T^{**}f^{**}\| \le C\|f^{**}\|$ for all $f^{**} \in B^{**}$.

Theorem 5.6 *Let B be a homogeneous Banach space. Then*

$$\mathrm{lip}_B(\lambda)^{**} \cong \mathrm{Lip}_B(\lambda)$$

with an isometric isomorphism.

We split the proof into several parts. Notation and definitions once introduced in one statement will be used in the following ones without reference. B shall always denote a homogeneous Banach space and we always assume that $0 < \lambda < 1$.

For $\varepsilon > 0$, we consider a sequence $d = \{d_n\}_{n\in\mathbb{N}_0}$ of type $[E]$ with $C_d \le 1+\varepsilon$. We let F_n $(n \in \mathbb{N})$ be the space B equipped with the norm

$$\|\cdot\|_{F_n} = \frac{1}{2^n}\,\varepsilon\|\cdot\|_B + \frac{w_B(\cdot, d_{n-1})}{(1-d_{n-1})^\lambda}.$$

We define

$$F_{d,\varepsilon} = B \oplus_1 \left(\sum\nolimits_{n\in\mathbb{N}} F_n\right)_0. \tag{5.2}$$

Here $\oplus_1$ denotes the l_1-sum of the two Banach spaces. We denote elements in $F_{d,\varepsilon}$ by $\{f_n\}_{n\in\mathbb{N}_0}$, where $f_0 \in B$ and $\{f_n\}_{n\in\mathbb{N}} \in \left(\sum_{n\in\mathbb{N}} F_n\right)_0$.

Lemma 5.7 *Let $\varepsilon > 0$, and let the sequence $d = \{d_n\}_{n\in\mathbb{N}_0}$ be of type $[E]$ with $C_d \le 1+\varepsilon$. Then*

$$\phi_{d,\varepsilon}:\ \mathrm{Lip}_B(\lambda) \to F_{d,\varepsilon},\ f \mapsto (f, (f, ..., f, ...)),$$

is a topological isomorphism and, for all $f \in \mathrm{Lip}_B(\lambda)$,

$$\tfrac{1}{1+\varepsilon}\|f\|_{\mathrm{Lip}} \le \|\phi_{d,\varepsilon} f\|_{F_{d,\varepsilon}} \le (1+\varepsilon)\|f\|_{\mathrm{Lip}}. \tag{5.3}$$

The same is valid for the restriction of $\phi_{d,\varepsilon}$ to $\mathrm{lip}_B(\lambda)$, which we denote by $\varphi_{d,\varepsilon}$.

Proof. By Lemma 4.4 we have that $\frac{1}{1+\varepsilon}\ \|f\|_{\mathrm{Lip}} \le \|f\|_{\mathrm{Lip},d} \le \|f\|_{\mathrm{Lip}}$. Since $\|f\|_{\mathrm{Lip},d} \le \|\phi_{d,\varepsilon} f\|_{F_{d,\varepsilon}} \le (1+\varepsilon)\|f\|_{\mathrm{Lip},d}$, we get that (5.3) holds true. □

We let

$$\pi_{d,\varepsilon} : F_{d,\varepsilon} \to B,\ (f_0, (f_1, ..., f_n, ...)) \mapsto f_0.$$

We identify the dual and second dual of $F_{d,\varepsilon}$ in the way pointed out above.

Lemma 5.8 *For the second adjoint of $\varphi_{d,\varepsilon}$, we have that*

$$\mathrm{im}(\varphi_{d,\varepsilon}^{**}) = \{(Qf, (..., Qf, ...)) :\ f \in \mathrm{Lip}_B(\lambda)\}. \tag{5.4}$$

Proof. Via identification, we get $F_{d,\varepsilon}^{**} = B^{**} \oplus_1 \left(\sum_{n\in\mathbb{N}} F_n^{**}\right)_\infty$. We let F_{const} be the constant sequences in $B^{**} \oplus_1 \left(\sum_{n\in\mathbb{N}} F_n^{**}\right)_\infty$. We start out by showing that $\mathrm{im}(\varphi_{d,\varepsilon}^{**}) \subset F_{\mathrm{const}}$. To this end we show that F_{const} is weak-$*$-closed. Let accordingly $x^{(i)**} \to x^{**}$ be a weak-$*$-convergent net in F_{const} with limit x^{**}. For $f^* \in B^*$, and $n_1, n_2 \in \mathbb{N}$ with $n_1 \ne n_2$, we choose two one-peak-sequences $\{x_n^*\}_{n\in\mathbb{N}_0}, \{y_n^*\}_{n\in\mathbb{N}_0} \in B^* \oplus_\infty \left(\sum_{n\in\mathbb{N}_0} F_n^*\right)_1$

such that $\{x_n^*\}_{n\in\mathbb{N}_0} = \{\delta_{nn_1} f^*\}_{n\in\mathbb{N}_0}$ and $\{y_n^*\}_{n\in\mathbb{N}_0} = \{\delta_{nn_2} f^*\}_{n\in\mathbb{N}_0}$. Here δ is the Kronecker symbol. We get that

$$\begin{aligned} x_{n_1}^{**} f^* = \lim_{i\in I} x_{n_1}^{(i)**} f^* \;&=\; \lim_{i\in I} x^{(i)**}\{x_n^*\}_{n\in\mathbb{N}_0} \;= \\ &=\; \lim_{i\in I} x^{(i)**}\{y_n^*\}_{n\in\mathbb{N}_0} \;=\; \lim_{i\in I} x_{n_2}^{(i)**} f^* \;=\; x_{n_2}^{**} f^*. \end{aligned}$$

This means that x^{**} is constant. Hence F_{const} is weak-$*$-closed. Let j denote the embedding of $\mathrm{lip}_B(\lambda)$ into B. Then $j = \pi_{d,\varepsilon} \circ \varphi_{d,\varepsilon}$, and it follows that $j^{**} = \pi_{d,\varepsilon}^{**} \circ \varphi_{d,\varepsilon}^{**}$. Since $\mathrm{im}(\varphi_{d,\varepsilon}^{**}) \subset F_{\text{const}}$, we know that $\pi^{**}_{|\,\mathrm{im}(\varphi_{d,\varepsilon}^{**})}$ is injective and so,

$$\begin{aligned} \mathrm{im}(\varphi_{d,\varepsilon}^{**}) &= \pi^{**-1}_{|\,\mathrm{im}(\varphi_{d,\varepsilon}^{**})}(j^{**}(\mathrm{lip}_B(\lambda))) \;\subset\; \pi^{**-1}_{|\,\mathrm{im}(\varphi_{d,\varepsilon}^{**})}(Q(B)) \\ &\subset \{(Qf, (..., Qf, ...)) :\; f \in B,\; \|f\|_B + \sup_{n\in\mathbb{N}} \|f\|_{F_n} < \infty\} \\ &= \{\{Qf\}_{n\in\mathbb{N}_0} :\; f \in \mathrm{Lip}_B(\lambda)\}, \end{aligned}$$

where the first inclusion follows from Proposition 5.3 and Proposition 5.5. Here $\{Qf\}_{n\in\mathbb{N}_0}$ is the sequence having Qf in every component. For the reverse inclusion in (5.4) let $f \in \mathrm{Lip}_B(\lambda)$. Choose a positive polynomial approximation kernel $\{K_m\}_{m\in\mathbb{N}}$ and define $f^{(m)} := K_m * f$. Then $f^{(m)} \to f$ in B and $f^{(m)} \in \mathrm{lip}_B(\lambda)$, since $f^{(m)}$ is a polynomial. Furthermore, $\|f^{(m)}\|_{\mathrm{Lip}_B(\lambda)} \le \|f\|_{\mathrm{Lip}_B(\lambda)}$. We intend to show that $\varphi_{d,\varepsilon} f^{(m)}$ is weak-$*$-convergent to $\{Qf\}_{n\in\mathbb{N}_0}$, which would complete the proof. To that end let $f^* = \{f_i^*\}_{i\in\mathbb{N}_0} \in B \oplus_\infty \left(\sum_{i\in\mathbb{N}} F_i^*\right)_1$ and $\delta > 0$. Choose $K \in \mathbb{N}$ such that

$$\sum_{i=K+1}^{\infty} \|f_i^*\|_{F_i} \le \frac{\delta}{4 \cdot \|\phi_{d,\varepsilon} f\| \cdot (1+\varepsilon)^2}.$$

Because F_i is topologically isomorphic to B for all $i \in \mathbb{N}$ we can find $M \in \mathbb{N}$ such that for all $i \in \{1, \dots, K\}$, we have

$$\|f^{(m)} - f\|_B + \|f^{(m)} - f\|_{F_i} \le \frac{\delta}{4 \cdot \|f^*\|} \quad \text{for all} \;\; m \ge M.$$

We obtain, for $m \ge M$,

$$\begin{aligned} &\{Qf\}_{n\in\mathbb{N}_0}(f^*) - (Q\varphi_{d,\varepsilon} f^{(m)})(f^*) = \left|\sum\nolimits_{i=0}^{\infty} f_i^*(f) - \sum\nolimits_{i=0}^{\infty} f_i^*(f^{(m)})\right| \\ &\le \left|\sum\nolimits_{i=K+1}^{\infty} f_i^*(f)\right| + \left|\sum\nolimits_{i=0}^{K} f_i^*(f - f^{(m)})\right| + \left|\sum\nolimits_{i=K+1}^{\infty} f_i^*(f^{(m)})\right|. \end{aligned}$$

We estimate the first summand by

$$\left|\sum\nolimits_{i=K+1}^{\infty} f_i^*(f)\right| \le \left(\sum\nolimits_{i=K+1}^{\infty} \|f_i^*\|_{F_i}\right) \sup_{i\ge K+1} \;\|f\|_{F_i} \le \tfrac{\delta}{4\,\|\phi_{d,\varepsilon} f\|} \|\phi_{d,\varepsilon} f\| \le \tfrac{\delta}{4},$$

the second summand by

$$\left|\sum\nolimits_{i=0}^{K} f_i^*(f - f^{(m)})\right| \le \tfrac{\delta}{2\cdot\|f^*\|} \cdot \|f^*\| \le \tfrac{\delta}{2},$$

and the last summand using (5.3) by

$$\begin{aligned} \left|\sum\nolimits_{i=K+1}^{\infty} f_i^*(f^{(m)})\right| &\le \left(\sum\nolimits_{i=K+1}^{\infty} \|f_i^*\|_{F_i}\right) \sup_{K+1\le i} \|f^{(m)}\|_{F_i} \\ &\le \tfrac{\delta}{4\|\phi_{d,\varepsilon} f\|} \tfrac{1}{(1+\varepsilon)^2} \|\varphi_{d,\varepsilon} f^{(m)}\| \le \tfrac{\delta}{4}. \end{aligned}$$

Altogether, $\{Qf\}_{n\in\mathbb{N}_0}(f^*) - (Q\varphi_{d,\varepsilon} f^{(m)})(f^*) \le \delta$. This completes the proof. □

Proposition 5.9 *The mapping*

$$j^{**} : \mathrm{lip}_B(\lambda)^{**} \to \mathrm{Lip}_B(\lambda)$$

is an isometric isomorphism.

Proof. Let $f^{**} \in \mathrm{lip}_B(\lambda)^{**}$ and $\varepsilon > 0$. Choose $F_{d,\varepsilon}$ according to Lemma 5.7, such that $\frac{1}{1+\varepsilon}\|g\| \leq \|\phi_{d,\varepsilon} g\| \leq (1+\varepsilon)\|g\|$ for all $g \in \mathrm{Lip}_B(\lambda)$. In particular, these inequalities are valid for $\mathrm{lip}_B(\lambda)$. By the properties of second adjoints, we obtain that $\frac{1}{1+\varepsilon}\|f^{**}\| \leq \|\varphi^{**}_{d,\varepsilon} f^{**}\| \leq (1+\varepsilon)\|f^{**}\|$. If we denote the right hand space of (5.4) by Y and consider $\phi^{-1}_{d,\varepsilon} : Y \to \mathrm{Lip}_B(\lambda)$, $y \mapsto \phi^{-1}_{d,\varepsilon} y$, we get by using (5.3) that, for all $y \in Y$, $\frac{1}{(1+\varepsilon)}\|y\| \leq \|\phi^{-1}_{d,\varepsilon} y\| \leq (1+\varepsilon)\|y\|$. Furthermore, $j^{**} = \phi^{-1}_{d,\varepsilon} \circ \varphi^{**}_{d,\varepsilon}$, and it follows that

$$\tfrac{1}{(1+\varepsilon)^2}\|f^{**}\| \leq \|j^{**} f^{**}\| \leq (1+\varepsilon)^2 \|f^{**}\|.$$

Since ε has been arbitrary, the proof is complete. □

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