manuscript submitted to Advances in Computational Mathematics
11/2008, revised version 08/2009

Johannes Wallner · Esfandiar Nava Yazdani · Andreas Weinmann

Convergence and smoothness analysis of subdivision rules in Riemannian and symmetric spaces

Abstract After a discussion on definability of invariant subdivision rules we discuss rules for sequential data living in Riemannian manifolds and in symmetric spaces, having in mind the space of positive definite matrices as a major example. We show that subdivision rules defined with intrinsic means in Cartan-Hadamard manifolds converge for all input data, which is a much stronger result than those usually available for manifold subdivision rules. We also show weaker convergence results which are true in general but apply only to dense enough input data. Finally we discuss C^1 and C^2 smoothness of limit curves.

1 Introduction

Several subdivision rules for manifold-valued data have been proposed and successfully analyzed in recent years. *Intrinsic* constructions work by replacing certain elementary constructions which apply to vector space data by analogous ones which operate on manifold data. Examples of operations thus modified are binary affine averages [11,15], or point-vector addition [13,4]. Other, *extrinsic* methods perform linear subdivision and add a projection afterwards [15,16]. The present paper is concerned with intrinsic subdivision rules and their properties. In particular, we discuss the following:

– Geometries which can possibly support meaningful subdivision (in the sense of invariance with respect to a defining transformation group);

J. Wallner and A. Weinmann
Institut für Geometrie, Technische Universität Graz. Kopernikusgasse 24, A 8010 Graz, Austria. E-mail: {j.wallner,andreas.weinmann}@tugraz.at

E. Nava Yazdani
Drexel University, Philadelphia, U.S.A. E-mail: navayazd@gmx.de

- Subdivision defined by the Riemannian center of mass (intrinsic mean) in Riemannian and symmetric spaces, and also log/exp subdivision in such spaces;
- Situations where convergence occurs for *all* input data (namely in Cartan-Hadamard manifolds for positive mask).
- Convergence analysis which applies to 'dense enough' input data, and further, C^1 and C^2 smoothness analysis of limit curves, provided they exist. These results are of the kind which have been obtained before.
- Discussion of subdivision in the set of positive definite symmetric matrices, this data type being relevant for example for the processing of diffusion tensor images.

We restrict ourselves to univariate subdivision rules defined by

$$p = (p_i)_{i\in\mathbb{Z}} \implies Sp = (Sp_i)_{i\in\mathbb{Z}} \text{ where } Sp_i = \sum\nolimits_{j\in\mathbb{Z}} a_{i-Nj} p_j. \qquad (1)$$

Here $N \in \{2, 3, \dots\}$ is the dilation factor of the rule, and the *mask* $(a_j)_{j\in\mathbb{Z}}$ has only finitely many nonzero coefficients. We require the affine invariance of the subdivision rule, which means $\sum_j a_{i-Nj} = 1$ for all i. This implies that S has a derived rule S^* with $S^*\Delta = N\Delta S$, where $\Delta p_i = p_{i+1} - p_i$.

Meaningful geometric constructions

Like any other construction, subdivision in a certain geometry should be meaningful, which means being *invariant* with respect to the transformations which are constitutive to that geometry. For example, a subdivision rule T for points p_i on the unit sphere should be invariant with respect to any rotation g in the sense that $T(g \circ p) = g \circ Tp$. Likewise, a subdivision rule for straight lines in space should be invariant with respect to the Euclidean motion group SE_3.

Even simple concepts like the midpoint of two geometric entities are sometimes not invariantly definable. An example of this behaviour is furnished by the projective space $\mathbb{R}P^n$, where any triple x, y, m of collinear points can be mapped into any other triple x', y', m' of collinear points by a projective transformation. The same is true for triples of non-collinear points. It follows that there exists *no concept of midpoint* in projective space in the sense of a function $m(x, y)$ such that for all projective transformations g, we have $g(m(x, y)) = m(g(x), g(y))$. The following paragraph shows that computing midpoints is actually a special case of subdivision, and so we conclude that such projective spaces do not admit invariant subdivision rules.

For computing midpoints via subdivision, consider special input data p_i with $p_i = x$ for $i < 0$ and $p_i = y$ for $i \geq 0$. Letting $a = \frac{1}{8}(\dots, 0, 1, 4, 6, 4, 1, 0, \dots)$ in Equ. (1) results in the well known cubic Lane-Riesenfeld subdivision rule

$$Sp_{2i} = \frac{1}{2}\Big(\frac{3}{4}p_i + \frac{1}{4}p_{i-1}\Big) + \frac{1}{2}\Big(\frac{3}{4}p_i + \frac{1}{4}p_{i+1}\Big), \ Sp_{2i+1} = \frac{1}{2}p_i + \frac{1}{2}p_{i+1}. \qquad (2)$$

Apparently $Sp_1 = \frac{x+y}{2}$, so no invariant analogue of the cubic Lane-Riesenfeld rule exists in projective spaces. An analogous argument applies to other weighted averages and appropriate subdivision rules.

2 Geometric constructions in symmetric paces

2.1 Homogeneous spaces

The geometries which fit Felix Klein's *Erlangen program* are characterized by the action of a transformation group on them. They can be formally described by the concept of *homogeneous space*. We first discuss this in general and then specialize to the case of *symmetric spaces*. The latter turn out to possess just the right number of degrees of freedom in their transformation groups to make subdivision rules definable. For the reader's convenience we illustrate the abstract definitions which follow below by the example Pos_n.

Assume that the group G acts on on the set X as a transformation group: for any $g \in G$, the mapping $x \mapsto g \circ x$ maps X into X such that $e \circ x = x$ and $(gh) \circ x = g \circ (h \circ x)$). We further require that for given x and y there is some g with $g \circ x = y$. We choose a *base point b* and identify a point $x \in X$ with the set of transformations which map the base to x: This means that with $\pi(g) = g \circ b$, x is identified with $\pi^{-1}x$. Obviously $\pi^{-1}x$ is a set of type $g \cdot K$, where g is any element of $\pi^{-1}x$ and K consists of those $g \in G$ with $g \circ b = b$. We can therefore use the notation $G/K = \{gK \mid g \in G\}$ instead of X. If now a point of X is written as "gK", then $g \circ hK = (gh)K$.

Example 1 Pos_n is defined as the set of positive definite symmetric $n \times n$ matrices. The group GL_n acts on X via $g \circ x = gxg^T$. We choose the unit matrix as a base point, and so $K = \mathrm{O}_n$ and $\pi(g) = gg^T$. Identifying a matrix $x \in \mathrm{Pos}_n$ with the set of $g \in \mathrm{GL}_n$ such that $gg^T = x$, we have $\mathrm{Pos}_n = \mathrm{GL}_n/\mathrm{O}_n$.

We consider only the case that G is a Lie group, K is a Lie subgroup, and X is a smooth manifold. The Lie algebras of G, K are denoted by $\mathfrak{g}$, $\mathfrak{k}$, respectively. As $\pi^{-1}e = K$, the kernel of the differential $d\pi_e$ equals $\mathfrak{k}$.

Example 2 Continuing Example 1, we get $\mathfrak{g} = \mathfrak{gl}_n$ (the Lie algebra of $n \times n$ matrices) and $\mathfrak{k} = \mathfrak{so}_n$, which is the Lie algebra of skew-symmetric matrices.

2.2 Symmetric Spaces

The theory of symmetric spaces is very extensive (see for instance the classic [5]). For our purposes global properties of the space are not important, so we consider what Section IX.2 of [8] calls the infinitesimal version of a symmetric space. We call a homogeneous space *symmetric*, if and only if $\mathfrak{k}$ occurs as the +1 eigenspace of a reflection $s : \mathfrak{g} \to \mathfrak{g}$ which is also a Lie algebra automorphism — this means that $s^2 = \mathrm{id}$ and $s([v, w]) = [s(v), s(w)]$ for all v, w. The -1 eigenspace of s is denoted by $\mathfrak{s}$.

Remark 1 This definition is more general than the more common definition which requires that K is (an open subgroup of) the fixed point set of a Lie group automorphism $\sigma : G \to G$ with the property $\sigma^2 = \mathrm{id}$, the reflection s being the differential of σ. If G is simply connected, these notions coincide, as for any s we can find an appropriate σ whose differential is s.

Example 3 We see that $s(v) = -v^T$ makes Pos_n a symmetric space, because $s(vw - wv) = s(v)s(w) - s(w)s(v)$, and the $+1$ eigenspace of s equals $\mathfrak{k} = \mathfrak{so}_n$. The -1 eigenspace is $\mathfrak{s} = \mathrm{Sym}_n$ (i.e., the space of symmetric $n \times n$ matrices). Apparently, the mapping $\sigma(g) := (g^{-1})^T$ is an automorphism of the group GL_n which has O_n as its fixed point set. It further obeys $\sigma^2 = \mathrm{id}$ and $d\sigma = s$. Thus, Pos_n is a symmetric space not only in the infinitesimal sense, but also in the narrower sense.

A transformation $x \mapsto g \circ x$ does not only map points, but also tangent vectors, via its differential: If $x(t)$ is a curve with $\frac{d}{dt}x(0) = v$, we denote the derivative $\frac{d}{dt}(g \circ x(t))|_{t=0}$ by the symbol $g \circ v$. We would also like to represent tangent vectors of X in terms of the groups G, K: As $\mathfrak{s}$ and $\mathfrak{k}$ are complementary subspaces, the restriction of the differential $d\pi_e$ to the subspace $\mathfrak{s}$ is 1-1 and onto. We can therefore uniquely represent a tangent vector v attached to the base point by a vector $\tilde{v} \in \mathfrak{s}$.

Example 4 In the setting of Example 1, the assumption $y(t) = g \circ x(t) = gx(t)g^T$ with $\frac{d}{dt}x = v$ implies that $\frac{dy}{dt} = g\frac{dx}{dt}g^T$, i.e., $g \circ v = gvg^T$. As to representing tangent vectors of Pos_n, we first consider the differential of the projection: The computation $\frac{d}{dt}\big|_{t=0}\pi(e + t\tilde{v}) = \tilde{v} + \tilde{v}^T$ implies that $d\pi_e(\tilde{v}) = \tilde{v} + \tilde{v}^T$. The tangent vector space of Pos_n at e (and indeed in any point) equals Sym_n. Obviously $v \in \mathrm{Sym}_n$ is represented by $\tilde{v} = v/2 \in \mathrm{Sym}_n$, because then $d\pi_e(\tilde{v}) = v$.

We consider symmetric spaces for two reasons: One is that geometric constructions relevant to subdivision processes can be consistently defined in them. Another reason is that several prominent geometries fall into this category – we have already encountered Pos_n. Others are the sets of unit vectors of $\mathbb{R}^{n+1}$ (leading to $S^n = \mathrm{O}_{n+1}/\mathrm{O}_n$), or the set of d-dimensional linear subspaces of $\mathbb{R}^n$ (which is called the Grassmannian $G_{d,n}$).

2.3 The exponential mapping.

For the purpose of transferring the definition of subdivision rules from Euclidean space to a more general setting, operations $v = y \ominus x$ and $y = x \oplus v$ which are analogous to the difference vector of points and the sum of point and vector are very useful [13]. We describe how to invariantly define such operations in Riemannian manifolds and in symmetric spaces by means of the exponential mapping .

This concept has already been employed for the purpose of subdivision in Lie groups, where it makes sense to let $y \ominus x = \log(x^{-1}y)$ and $x \oplus v =$

$x \exp(v)$, where exp is the exponential mapping in the group [13,2,4,18]. In Riemannian geometry a similar construction is $x \oplus v := \exp_x(v)$ and $y \ominus x = \exp_x^{-1}(y)$. Here $\exp_x(v)$ is the exponential mapping which follows the geodesic line emanating from x in direction v for the amount of arc length given by $\|v\|$.

The case of symmetric spaces is similar to the Riemannian case. Most symmetric spaces which occur in the literature also have a Riemannian exponential mapping, coinciding with the one defined below. There are, however, symmetric spaces which do not admit an invariant Riemannian metric, not even a pseudo-Riemannian metric. Examples are furnished by the symmetric space of d-dimensional affine subspaces of $\mathbb{R}^n$, if $d \neq 0$ and $(d, n) \neq (1, 3)$.

We define:

Definition 1 *Assume that in the symmetric space $X = G/K$ a vector v is attached to the base point b, and $w = g \odot v$ is attached to $x = g \circ b$. If $\tilde{v} \in \mathfrak{s}$ represents v, then* $\mathrm{Exp}_x(w) = \mathrm{Exp}_{g \circ b}(g \odot v) := g \exp(\tilde{v}) \circ b$.

The choice of g such that $g \circ b = x$ is not unique, but it is well known that $\mathrm{Exp}_x(w)$ does not depend on this choice.[1] Note that *Exp* is an *invariant* mapping which means that for any transformation $g \in G$, we have $\mathrm{Exp}_{g \circ x}(g \odot w) = g \circ \mathrm{Exp}_x(w)$.

Example 5 We want to compute $\mathrm{Exp}_x(w)$ in the space $\mathrm{Pos}_n = \mathrm{GL}_n/\mathrm{O}_n$. We first find $g \in \mathrm{GL}_n$ such that $g \circ b = x$. This is done by letting $g = x^{1/2}$, itself contained in Pos_n. Solving for v such that $g \odot v = w$ results in $v = x^{-1/2} w x^{-1/2}$, which in $\mathfrak{s}$ is represented by $\tilde{v} = \frac{1}{2} v$. We evaluate $\mathrm{Exp}_x(w) = g \exp(v) \circ b = g \exp(\tilde{v})^2 g$ and observe that $\exp(aba^{-1}) = a(\exp b)a^{-1}$. This yields

$$\mathrm{Exp}_x(w) = x^{1/2} \exp(x^{-1/2} w x^{-1/2}) x^{1/2} = x \exp(x^{-1} w).$$

With the exponential mapping we can now define $\oplus$ and $\ominus$ by letting

$$x \oplus w = \mathrm{Exp}_x(w), \quad y \ominus x = \mathrm{Exp}_x^{-1}(y). \tag{3}$$

While $\oplus$ is globally defined, this is generally not the case for $\ominus$ (but see the remarks on well-definedness below). The following is well known [5]:

Lemma 2 *The differential of the mapping $v \mapsto x \oplus v$ at $v = 0$ equals the identity.*

Example 6 In Pos_n, $\ominus$ is globally defined, because Example 5 implies $y \ominus x = x^{1/2} \log(x^{-1/2}\, y\, x^{-1/2})\, x^{1/2}$, and $\log|_{\mathrm{Pos}_n}$ is well defined.

[1] The following proof is similar to the proof in [5, p. 209f]: By construction, $k \in K$ implies that $\pi(kgk^{-1}) = \pi(kg) = k \circ \pi(g)$, and so by a limit $g(t) \to e$ with $dg/dt = v$ we have $d\pi(\mathrm{Ad}_k v) = k \odot d\pi(v)$. If $v \in \mathfrak{s} = T_b X$, we can even write $\mathrm{Ad}_k v = k \odot v$. Now we consider two group elements $g, h \in G$ which transform the base point to the point x under consideration and construct $\mathrm{Exp}_x(tw)$ in two ways: $x = g \circ b = h \circ b \implies k := h^{-1} g \in K$; $w = g \odot u = h \odot v$ for $u, v \in \mathfrak{s} \implies v = k \odot u = \mathrm{Ad}_k(u) \implies h \circ \mathrm{Exp}_b(tv) = \pi(h \exp(t \,\mathrm{Ad}_k u)) = \pi(gk^{-1} k \exp(tu)) = g \circ \mathrm{Exp}_b(tu)$. Here we have used that $\exp(\mathrm{Ad}_k v) = k(\exp v)k^{-1}$.

2.4 Weighted binary averages.

The binary average defined by

$$\text{g-av}_\alpha(x, y) = x \oplus \alpha(y \ominus x) \tag{4}$$

is analogous to the affine average $(1-\alpha)x+\alpha y = x+\alpha(y-x)$. In Riemannian and in symmetric spaces, the curves $t \mapsto x \oplus (tv)$ are called geodesics. Thus this average should be called the geodesic average, as it lies on a geodesic which connects x, y. The relations $\text{g-av}_t(x, y) = \text{g-av}_{1-t}(y, x)$, $\text{g-av}_{1/2}(x, y) = \text{g-av}_{1/2}(y, x)$ follow from well known properties of *Exp* (see [5]).

Example 7 In Pos_n, the geodesic average is expressed as $\text{g-av}_\alpha(x, y) = x \cdot \exp(\alpha \log(x^{-1}y)) = x \cdot (x^{-1}y)^\alpha$.

2.5 Intrinsic means.

The weighted mean $x^* = \sum \alpha_j x_j$ of points $x_j \in \mathbb{R}^d$ with weights α_j summing up to 1 is quite obviously characterized as a minimizer:

$$\nabla f_\alpha(x^*) = 0 \text{ where } f_\alpha(x) = \sum\nolimits_{j=1}^{n} \alpha_j \operatorname{dist}(x_j, x)^2. \tag{5}$$

Further, if for a moment we adopt the notation $x \ominus y$ for the ordinary difference $x - y$, x^* is also uniqely characterized by the balance condition

$$x^* = \text{mean}((x_1, \alpha_1), \dots, (x_n, \alpha_n)) \iff \sum\nolimits_{j=1}^{n} \alpha_j(x_j \ominus x^*) = 0. \tag{6}$$

Equations (5) and (6) have a meaning in Riemannian geometry, too. It is known that x^*, if defined by either equation, exists uniquely if we restrict ourselves to data which lie in small sets, and that then (5), (6) are equivalent (see [6,7] for more details). Globally however, x^* is in general not unique.

In any geometry where $\ominus$ is defined, (6) makes sense, and we would like to call any point x^* defined by (6) the *weighted intrinsic mean*. Its computation is usually possible only numerically. For two points ($n = 2$), the intrinsic mean reduces to the previously defined binary average.

2.6 Well-definedness of geometric constructions

If a geometric construction of Euclidean space is extended to a more general setting, it frequently turns out to be no longer globally defined. For example, the geodesic midpoint in surfaces might not exist (for incomplete surfaces), or might not exist uniquely (for example for antipodes on the sphere). The same is true for the more general intrinsic mean construction.

For this reason, general statements on existence, convergence, etc. of subdivision rules in nonlinear geometries can be true only for *dense enough* input data (see for instance the discussion in the first paper on this subject, namely [15]). In some geometries relevant to applications we can be more specific, and $\ominus$ may even be globally well defined. This is the case of the space Pos_n, as seen from Example 6. The well-definedness of the intrinsic mean is the subject of Section 4.

3 Defining subdivision in Riemannian and symmetric spaces

One way to transfer the definition of a linear subdivision rule S as given by (1) to a more general setting is to rewrite it in the form

$$Sp_{Ni+k} = \sum\nolimits_{j\in\mathbb{Z}} a_{k+N(i-j)} p_j = m_{i,k} + \sum\nolimits_{j\in\mathbb{Z}} a_{k+N(i-j)}(p_j - m_{i,k}), \tag{7}$$

where $m_{i,k}$ are arbitrary except for $Ni + k = Ni' + k' \implies m_{i,k} = m_{i',k'}$. We now replace every occurrence of + and − by the operators $\oplus$ and $\ominus$:

$$Tp_{Ni+k} = m_{i,k} \oplus \sum\nolimits_{j\in\mathbb{Z}} a_{k+N(i-j)}(p_j \ominus m_{i,k}) \quad (k = 0, \dots, N-1). \tag{8}$$

This rule (the *log-exponential analogue* of S) operates on sequences of input data which live in a Lie group, in a symmetric space or in a Riemannian manifold. It was essentially first considered by D. Donoho (see also [13]). As to the choice of the points $m_{i,k}$, we could let $m_{i,k} = p_i$ or $m_{i,k} = \text{g-av}_{1/2}(p_i, p_{i+1})$. Smoothness of such rules in Lie group rules is analyzed for instance by [4, 18].

Using the intrinsic mean, we can directly convert Equation (1) into an invariant subdivision rule. We use one of the two equivalent definitions

$$Tp_i = \mathrm{mean}((p_j, a_{i-Nj})_{j\in\mathbb{Z}}), \text{ or } Tp_{Ni+k} = \mathrm{mean}((p_j, a_{Ni+k-Nj})). \tag{9}$$

We call T the *i-mean analogue* of the linear rule S. Interestingly, (9) becomes a special case of (8), if we choose $m_{i,k} = \mathrm{mean}((p_j, a_{Ni+k-Nj}))$, since then the expression $\sum a_{k+N(i-j)}(p_j \ominus m_{i,k})$ in (8) equals 0.

Any convergent linear rule (which in generally accepted usage does not include the trivially convergent ones with spectral radius less than one) consists of affine combinations and can be expressed in terms of binary affine averages [15]. Sometimes this expression is very symmetric and thus a candidate for replacing affine averages by geodesic averages. For instance, a *geodesic analogue* of the Lane-Riesenfeld rule of Equation (2) reads

$$Tp_{2i} = \text{g-av}_{\frac{1}{2}}\Big(\text{g-av}_{\frac{1}{4}}(p_i, p_{i-1}), \text{g-av}_{\frac{1}{4}}(p_i, p_{i+1})\Big),\ Tp_{2i+1} = \text{g-av}_{\frac{1}{2}}(p_i, p_{i+1}).$$

4 Global convergence analysis

In general, convergence statements for nonlinear subdivision rules are only possible for dense enough input data. Special geometries, however, allow us to give more precise statements. We postpone the general case to the next section and concentrate on Cartan-Hadamard (CH) manifolds, which are defined by two conditions: (i) the sectional curvature is nonpositive and

(ii) simple connectedness. The space Pos_n, which from the viewpoint of differential geometry is studied in chapter XII of [9], is our major exampe. The intrinsic mean defined by (6) is unique in CH manifolds, as shown by Th. 9.1 in Ch. 8 of [8].

Lemma 3 *Consider a CH manifold M and points $x_1, \dots, x_r \in M$. Define the intrinsic mean x^* by nonnegative weights $(\alpha_j)_{j=1}^r$ with $\sum \alpha_j = 1$. Similarly x^{**} is defined by weights β_j. Then the Riemannian distance of x^*, x^{**} obeys*

$$\operatorname{dist}(x^*, x^{**}) \leq \sum_j |\sigma_j - \tau_j| \cdot \max_{1\leq k\leq r} \operatorname{dist}(x_k, x_{k+1}), \text{ with } \sigma_j = \sum_{i\leq j} \alpha_i, \ \tau_j = \sum_{i\leq j} \beta_i.$$

Proof We consider σ as a function $\sigma : \{1, \dots, r\} \to \mathbb{R}$ and define $\sigma^{-1}(t) = \sup\{j \mid \sigma_j < t\}$. By construction, σ^{-1} maps a subinterval of length α_j of $[0,1]$ to the index j, for all $j \in \{1, \dots, r\}$. Thus the function $f_\alpha(x)$ of Equ. (5) equals

$$f_\alpha(x) = \sum \alpha_j \operatorname{dist}(x_j, x)^2 = \int_{[0,1]} \operatorname{dist}(x_{\sigma^{-1}(t)}, x)^2 \, dt.$$

Analogously we convert f_β into an integral. This notation for f_α and f_β is the one of [6]. Theorem 1.5.1 and the proof of Cor. 1.6 of that paper show that

$$\operatorname{dist}(x^*, x^{**}) \leq \int_{[0,1]} \| \exp_{x^*}^{-1}(x_{\sigma^{-1}(t)}) - \exp_{x^*}^{-1}(x_{\tau^{-1}(t)}) \| \, dt.$$

The exponential mapping does not decrease distances in CH manifolds, so

$$\operatorname{dist}(x^*, x^{**}) \leq \int_{[0,1]} \operatorname{dist}(x_{\sigma^{-1}(t)}, x_{\tau^{-1}(t)}) \, dt \leq \sup_j \operatorname{dist}(x_j, x_{j+1}) \cdot \int_{[0,1]} |\sigma^{-1} - \tau^{-1}|.$$

The latter integral obviously equals $\sum |\sigma_j - \tau_j|$, so the proof is complete. □

Proposition 4 *Suppose that T is the i-mean analogue in a CH manifold of the linear rule S with dilation factor N and nonnegative mask coefficients a_j. Let*

$$\sigma_j^{(r)} = \sum\nolimits_{i\leq j} a_{r-Ni}, \quad \mu = \max\nolimits_{0\leq r\leq N} \sum\nolimits_j |\sigma_j^{(r+1)} - \sigma_j^{(r)}|. \tag{10}$$

If $\mu < 1$, then T converges to a continuous limit $T^\infty p$ for all input data p.

Proof Each point $T^k p_j$ is the intrinsic mean of points $T^{k-1} p_i$ with weights a_{j-Ni}. As only finitely many coefficients of the mask are nonzero, it therefore lies in the geodesic convex hull $C_{i,k}$ of a fixed number of data points $T^{k-1} p_i$. By Lemma 3,

$$\operatorname{dist}(T^k p_{i+1}, T^k p_i) \leq \mu^k \sup\nolimits_j \operatorname{dist}(p_j, p_{j+1}).$$

It follows that the diameter of $C_{i,k}$ is contracting with μ^k. We now interpolate the vertices $T^k p_i$ by a broken geodesic $c^{(k)} : \mathbb{R} \to M$, where $c^{(k)}|_{[i/N^k,(i+1)/N^k]}$ is the geodesic spanned by $T^k p_i$ and $T^k p_{i+1}$. Contraction of $C_{i,k}$ implies that for any interval $I = [a, b]$, $(c^{(k)}\big|_I)_{k\geq 0}$ is a Cauchy sequence in $C(I, M)$, when equipped with the metric $\operatorname{dist}(c, d) = \max_{t\in I}(\operatorname{dist}(c(t), d(t)))$. This metric is complete, so $T^\infty p$ exists. □

Theorem 5 *Let T be the i-mean analogue in a CH manifold of the linear rule S with dilation factor N. If the derived scheme S^* obeys $\|S^*\| < N$, then T converges to a continuous limit $T^\infty p$ for all input data p.*

Proof Consider special input data $q : \mathbb{Z} \to \mathbb{R}$ with $q_j = -1$ for $j \leq 0$ and $q_j = 0$ otherwise. Denote the mask of the derived scheme by $(a_j^*)_{j\in\mathbb{Z}}$. Then

$$\frac{1}{N}a_l^* = \frac{1}{N}S^*\Delta q_l = \Delta S q_l = Sq_{l+1} - Sq_l = \sum\nolimits_{k\leq 0}(-a_{l+1-Nk} + a_{l-Nk})$$
$$\implies \frac{1}{N}a_{r-Nj}^* = \sum\nolimits_{k\leq 0}(a_{r-N(j+k)} - a_{r+1-N(j+k)}) = \sum\nolimits_{i\leq j}(a_{r-Ni} - a_{r+1-Ni}).$$

In the terminology of Prop. 4, we have shown that $\frac{1}{N}a_{r-Nj}^* = \sigma_j^{(r)} - \sigma_j^{(r+1)}$. Thus

$$\sup\nolimits_r \sum\nolimits_j |\sigma_j^{(r)} - \sigma_j^{(r+1)}| = \frac{1}{N}\sup\nolimits_r \sum\nolimits_j |a_{r-Nj}^*| = \frac{1}{N}\|S^*\|.$$

We can thus directly apply Prop. 4, and the proof is complete. □

The condition that $\|S^*\| < N$ is rather weak. It is one of the easily checked sufficient conditions which ensure convergence of a linear scheme. It is very satisfactory that it occurs as a sufficient condition also in the nonlinear case.

Example 8 The i-mean analogue of all binary Lane-Riesenfeld subdivision schemes, starting with the piecewise linear one, converge for arbitrary input data in Pos_n, since we have $\|S^*\| = 1$ throughout.

5 C^1 smoothness analysis

In our analysis of subdivision rules in general Riemannian manifolds and symmetric spaces we invoke the general results presented in [15,14] and which give convergence only for dense enough input data. For both rules S, T, computing the i-th new data point is a function of ν of old data points, ν being globally bounded by the distribution of nonzero coefficients in the mask (a_i) of S. These two facts mean that the following theorems and their proofs are entirely local. Thus, we employ a local coordinate chart $\chi : X \to \mathbb{R}^n$ for the manifold X under consideration and denote the coordinate representations of $\oplus, T, \ldots$ by $\widetilde{\oplus}, \widetilde{T}, \ldots$, respectively. Another consequence of ν being bounded is that $\widetilde{T}$ has Taylor polynomials in the ordinary sense. The results is as follos:

Theorem 6 *Assume that the rule S with dilation factor N is defined by* (1)*, and that T is its geodesic analogue or log-exp analogue (in which case assume that the auxiliary points $m_{i,k}$ are chosen such that $m_{i,k} \to p_i$ as input data approach each other). If S is a convergent scheme, then T is convergent for dense enough input data. If in addition the derived scheme S^* is a convergent scheme (so S has C^1 limits), then also the limits of the nonlinear rule T enjoy C^1 smoothness.*

Note that Theorem 6 applies to the i-mean analogue of a linear rule, because it can be seen as a special case of the log-exp construction.

Proof Below we establish that the *proximity inequality* $\|\widetilde{T}x - Sx\|_\infty \le \text{const} \cdot \|\Delta x\|_\infty$ holds for dense enough input data. By [14, Lemma 3] this remains true if S, T are replaced by iterates S^k, T^k, respectively. Convergence to a continuous limit now follows from [15, Th. 3] applied to some iterate S^k, T^k such that $\|S^{*k}/N^k\| < 1$. Likewise, C^1 smoothness follows from [15, Th. 5] applied to S^k, T^k such that $\|S^{*k}/N^k\| < 1/N^{k/2}$. Such k exists by [1, Lemma 1].

As to the proximity inequality, Lemma 2 implies the Taylor linearizations $x \mathbin{\widetilde{\oplus}} v \doteq x + v$ as $v \to 0$ and $y \mathbin{\widetilde{\ominus}} x \doteq y - x$ as $y \to x$. We now aim at the first order Taylor polynomial of $\widetilde{T}(p)$, as input data p_i approach a constant sequence. As $m_{i,k}$ tends to that same sequence, the vectors $p_j \ominus m_{i,k}$ in (8) approach zero. It is therefore clear that the Taylor polynomial equals the linear rule S. It follows that $\widetilde{T}x_i = Sx_i + O(\|x_i - x_j\|^2)$, i.e., $\|\widetilde{T}x_i - Sx_i\| \le C \cdot \sup \|\Delta x_i\|^2$.

Note that the previous arguments includes the i-mean analogue. A similar argument is true if T is an analogue of S constructed from binary averages: The linearization $\widetilde{\text{g-av}}_t(x, y) \doteq (1-t)x + ty$ for $y \to x$ implies $\|\widetilde{\text{g-av}}_t(x, y) - (1-t)x - ty\| \le C'\|x - y\|^2$. By iteration, $\|\widetilde{T}x_i - Sx_i\| \le C'' \sup \|\Delta x_i\|^2$. □

6 C^2 smoothness analysis

We give conditions which ensure that the limit curves of subdivision rules in Riemannian and symmetric spaces enjoy C^2 smoothness, provided they exist. The proofs given here are similar to others which deal with Lie group data (cf. [4]). The recent manuscript [17] which shows general C^k smothness of a wide class of schemes does not cover the more general statement of our Theorem 7 (see Remark 2 below). In our method of proof, we follow [4]. The general result is as follows:

Theorem 7 *Consider the linear subdivision rule S and its log-exponential analogue T defined by* (8)*, including the i-mean analogue* (9) *as a special case. If S, S^*, and S^{**} are convergent schemes, then T's limits enjoy C^2 smoothness whenever they exist.*

Before we proceed with the proof we give some auxiliary results.

Lemma 8 *The composite operation $(x \mathbin{\widetilde{\oplus}} v \mathbin{\widetilde{\oplus}} w) \mathbin{\widetilde{\ominus}} x$ has the second order Taylor polynomial $v + w + \Psi_x(v, w)$ with bilinear Ψ_x, as $v, w \to 0$.*

Proof By Lemma 2, $(x \mathbin{\widetilde{\oplus}} v \mathbin{\widetilde{\oplus}} w) \mathbin{\widetilde{\ominus}} x \doteq v + w$ and therefore $(x \mathbin{\widetilde{\oplus}} v \mathbin{\widetilde{\oplus}} w) \mathbin{\widetilde{\ominus}} x \doteq\!\!\doteq v + w + \psi_x(v, w)$, where $\psi_x : \mathbb{R}^{2m} \to \mathbb{R}^m$ is homogeneous quadratic, having the general form

$$\psi_x(v, w) = \omega_{1x}(v, v) + \omega_{2x}(w, w) + \Psi_x(v, w)$$

with bilinear ω_{1x}, ω_{2x}, and Ψ_x. Now $(x \mathbin{\widetilde{\oplus}} v \mathbin{\widetilde{\oplus}} 0) \mathbin{\widetilde{\ominus}} x = (x \mathbin{\widetilde{\oplus}} v) \mathbin{\widetilde{\ominus}} x = v$ and $(x \mathbin{\widetilde{\oplus}} 0 \mathbin{\widetilde{\oplus}} w) \mathbin{\widetilde{\ominus}} x = (x \mathbin{\widetilde{\oplus}} w) \mathbin{\widetilde{\ominus}} x = w$ imply $\omega_{1x}(v, v) = 0$, $\omega_{2x}(w, w) = 0$. □

In the Lie group case, the expansion given of Lemma 8 reads $\log(\exp v \cdot \exp w) \doteq v + w + \frac{1}{2}[v, w]$. This similarity allows us to follow closely the proof of C^2 smoothness in [4]. The following lemma has been shown in more generality by [3]:

Lemma 9 *Assume that $\beta : V \times V \to V$ is bilinear, and $F : V^{\mathbb{Z}} \to V$ takes sequences as arguments, mapping them to $\sum A_{r,s}\beta(v_r, v_s)$, with only finitely many matrices $A_{r,s}$ being nonzero. Then F can be rewritten in the form*

$$F(v_j) = \sum\nolimits_{r,s\in\mathbb{Z}} (a_{r,s}\beta(v_r, \Delta^2 v_s) + b_{r,s}\beta(\Delta^2 v_r, v_s),$$

if and only if the generating Laurent polynomial $A(\xi, \eta) := \sum_{r,s\in\mathbb{Z}} A_{rs}\xi^r\eta^s$ has the properties

$$A(1,1) = \frac{\partial A}{\partial \xi}(1,1) = \frac{\partial A}{\partial \eta}(1,1) = \frac{\partial^2 A}{\partial \xi \partial \eta}(1,1) = 0.$$

Proof (of Theorem 7) As before, we use a tilde to indicate coordinate representations and let subdivision rules S and $\widetilde{T}$ act on data x_i living in $\mathbb{R}^n$. We introduce $v^l_{i,k} = x_l \mathbin{\widetilde{\ominus}} \widetilde{m}_{i,k}$, where $\widetilde{m}_{i,k}$ is the coordinate representation of the point $m_{i,k}$ of (8) (itself data dependent). With $(\widetilde{T} - S)x_{Ni+k} = F_{i,k}$ we have

$$F_{i,k} := \widetilde{m}_{i,k} \mathbin{\widetilde{\oplus}} \sum\nolimits_j a_{Ni-k-Nj} v^j_{i,k} - \widetilde{m}_{i,k} - \sum\nolimits_j a_{Ni+k-Nj}(\widetilde{m}_{i,k} \mathbin{\widetilde{\oplus}} v^j_{i,k} - \widetilde{m}_{i,k}).$$

Obviously, $F_{i,k} \doteq 0$, as as $v^j_i \to 0$. Aiming at a 2nd order Taylor polynomial of $\Delta(T - S)x_{Ni+k}$, we introduce $x \mathbin{\widetilde{\oplus}} v \doteq x + v + \phi_x(v, v)$, (with ϕ_x bilinear and symmetric). If neither $\widetilde{m}_{i,k} =: \widetilde{m}_i$ nor $v^j_{i,k} =: v^j_i$ depend on k, then

$$\begin{aligned} \Delta(\widetilde{T} - S)x_{Ni+k} = F_{i,k+1} - F_{i,k} \ \doteq\ & \phi_{\widetilde{m}_i}(\sum\nolimits_j a_{Ni+k+1-Nj} v^j_i, \text{ ''}\ ) \\ - \phi_{\widetilde{m}_i}(\sum\nolimits_j a_{Ni+k-Nj} v^j_i, \text{ ''}\ ) - \sum\nolimits_j & \Delta a_{Ni+k-Nj} \phi_{\widetilde{m}_i}(v^j_i, v^j_i). \end{aligned} \tag{11}$$

This computation is valid only for $k \in \{0, \ldots, N-1\}$. The following computation, with deals with the general case $\widetilde{m}_{i,k} \neq \widetilde{m}_{i,k+1}$, also includes the case $k = N-1$, by letting $\widetilde{m}_{i+1,0} = \widetilde{m}_{i,N}$.

Let $w_{i,k} = \widetilde{m}_{i,k+1} \ominus \widetilde{m}_i$ and use Lemma 8 to compute

$$\begin{aligned} x_j = \widetilde{m}_{i,k} \mathbin{\widetilde{\oplus}} v^j_{i,k} &= \widetilde{m}_{i,k+1} \mathbin{\widetilde{\oplus}} v^j_{i,k+1} = \widetilde{m}_{i,k} \mathbin{\widetilde{\oplus}} w_{i,k} \mathbin{\widetilde{\oplus}} v^j_{i,k+1} \\ &\doteq \widetilde{m}_{i,k} \mathbin{\widetilde{\oplus}} (w_{i,k} + v^j_{i,k+1} + \Psi_{\widetilde{m}_{i,k}}(w_{i,k}, v^j_{i,k+1})) \\ \implies v^j_{i,k} &\doteq w_{i,k} + v^j_{i,k+1} + \Psi_{\widetilde{m}_{i,k}}(w_{i,k}, v^j_{i,k+1}). \end{aligned}$$

Letting $u_{i,k} = \sum a_{N(i+1)-Nj} v^j_{i,k+1}$, this formula implies

$$\begin{aligned} u_{i,k} &\doteq \sum\nolimits_j a_{N(i+1)-Nj}(v^j_{i,k} - w_{i,k} - \Psi_{\widetilde{m}_{i,k}}(w_{i,k}, v^j_{i,k+1})) \\ &\doteq \sum\nolimits_j a_{N(i+1)-Nj} v^j_{i,k} - w_{i,k} - \Psi_{\widetilde{m}_{i,k}}(w_{i,k}, \sum\nolimits_j a_{N(i+1)-Nj} v^j_{i,k+1}). \end{aligned}$$

Note that the last term equals $\Psi_{\widetilde{m}_{i,k}}(w_{i,k}, u_{i,k})$. We continue with

$$\begin{aligned} Tx_{Ni+k+1} &= \widetilde{m}_{i,k} \mathbin{\widetilde{\oplus}} w_{i,k} \mathbin{\widetilde{\oplus}} u_{i,k} \doteqdot \widetilde{m}_{i,k} \mathbin{\widetilde{\oplus}} (w_{i,k} + u_{i,k} + \Psi_{\widetilde{m}_{i,k}}(w_{i,k}, u_{i,k})) \\ &\doteqdot \widetilde{m}_{i,k} + w_{i,k} + u_{i,k} + \Psi_{\widetilde{m}_{i,k}}(w_{i,k}, u_{i,k}) + \phi_{\widetilde{m}_{i,k}}(w_{i,k} + u_{i,k}, '') \\ &\doteqdot \widetilde{m}_{i,k} + \sum\nolimits_j a_{N(i+1)-Nj} v^j_{i,k} + \phi_{\widetilde{m}_{i,k}}(w_{i,k} + u_{i,k}, '') \\ &\doteqdot \widetilde{m}_{i,k} + \sum\nolimits_j a_{N(i+1)-Nj} v^j_{i,k} + \phi_{\widetilde{m}_{i,k}}(\sum\nolimits_j a_{N(i+1)-Nj} v^j_{i,k}, ''). \end{aligned}$$

The w_i's have disappeared and the result is the same as if $\widetilde{m}_{i,k+1} = \widetilde{m}_{i,k}$ — we get (11) again. In the following therefore without loss of generality we let $m_{i,k} = m_i$ and $v^j_{i,j} = v^j_i$.

Index shifts are irrelevant for the result, so we let $i = 0$ in what follows. Expanding (11), using symmetry and bilinearity of $\phi_{\widetilde{m}_i}$, yields

$$\begin{aligned} F_{0,k+1} - F_{0,k} &\doteqdot \sum\nolimits_{r,s} A_{r,s} \phi_{\widetilde{m}_0}(v^r_0, v^s_0), \quad \text{where} \\ A_{rs} &= a_{r+1-Ns} a_{r+1-Nr} - a_{r-Ns} a_{r-Nr} \text{ for } s \neq r, \\ A_{rs} &= a^2_{r+1-Ns} - a^2_{r-Ns} + (a_{r+1-Ns} - a_{r-Ns}) \text{ for } s = r. \end{aligned}$$

This is analogous to [4, Equ. (13)], where it is also shown that the $A_{r,s}$'s fulfill the conditions of Lemma 9. We conclude that

$$F_{0,k+1} - F_{0,k} \doteqdot \sum\nolimits_{r,s} \Big(a_{r,s} \phi_{\widetilde{m}_0}(v^r_0, \Delta^2 v^s_0) + b_{r,s} \phi_{\widetilde{m}_i}(\Delta^2 v^r_i, v^s_i) \Big).$$

The inequality $\|\phi(v, w)\| \leq \|\phi\| \cdot \|v\| \cdot \|w\|$ for any bilinear function and the cubic remainder term in the Taylor series now implies the existence of a constant C such that locally $\|F_{i,k+1} - F_{i,k}\|$ is bounded by $C \cdot \sup \|v^r_i\| \sup \|\Delta^2 v^r_i\| + O(3)$. The first order expansion $x \mathbin{\widetilde{\oplus}} y \doteq x + y$ shows that $v^r_i \doteq x^r - \widetilde{m}_i = (x_i - \widetilde{m}_i) + (x_{i+1} - x_i) + \cdots + (x_r - x_{r-1})$, so $\|v^r_i\| \leq C \cdot \sup_j \|\Delta x_j\| + O(2)$. The same argument shows that $\Delta^2 v^r_i \doteq \Delta^2 x^r_i$, so $\|\Delta^2 v^r_i\| \leq \sup_j \|\Delta^2 x_j\| + O(2)$. Summing up, we have the inequality

$$\|\Delta(S - \widetilde{T})x\| = \sup_{i \in \mathbb{Z}, 0 \leq k \leq N} \|F_{i,k+1} - F_{i,k}\| \leq C(\|\Delta x\|^3 + \|\Delta^2 x\| \, \|\Delta x\|),$$

as $v^j_i \to 0$. This is the general proximity condition of [14, Def. 4] which allows us to invoke [14, Theorem 6]. The required conditions that $\|S^*\| < N^{1/3}$ and $\|S^*\| \cdot \|S^{**}\| < N$ for S or one of its iterates are fulfilled by [1, Lemma 1]. We conclude that limit curves of T enjoy C^2 smoothness. □

Remark 2 The recent paper [17] shows C^k smoothness for a general class of manifold subdivision rules with dilation factor $N = 2$, where the auxiliary points $m_{i,k}$ of (8) are themselves chosen by an interpolatory subdivision rule. It seems obvious from numerical experiments that an arbitrary choice of auxiliary points $m_{i,k}$ destroys any smoothness higher than C^2 [17].

7 Examples

The space Pos_n of positive definite symmetric $n \times n$ matrices, already discussed in Examples 1–7, is of interest in vision and data processing, because Pos_n-valued data naturally occur for example in diffusion tensor imaging. Their 'geometric' (i.e., invariant) handling is the topic of several contributions, for instance [10]. It has various different interpretations, all of which lead to an exponential mapping. We shall see, however, that in fact all these different notions coincide, and that for showing properties of subdivision in Pos_n, we can apply Lie group results as well as results for Cartan-Hadamard manifolds.

7.1 Subdivision in Pos_n as a symmetric space.

It is shown in [9, Ch. XII] that Pos_n can be equipped with a Riemannian metric such that it becomes a Cartan-Hadamard manifold whose exponential mapping coincides with the one defined in Pos_n as a symmetric space by Example 5. We therefore have the following corollary of Theorem 5:

Corollary 10 *Let T be the i-mean analogue in Pos_n of the linear rule S with dilation factor N. If the derived scheme S^* obeys $\|S^*\| < N$, then T converges to a continuous limit $T^\infty p$ for all input data p.*

In all cases where Theorem 5 applies we can remove the assumption that the limit $T^\infty p$ exists from the statements of Theorems 6 and 7. The modified statements are formulated in the corollaries below, which in particular apply to Pos_n.

Corollary 11 *If S has positive mask and T is the i-mean analogue of S in a CH manifold, then all input data p generate limit curves $T^\infty p$ of C^1, provided $\|S^*\| < N$ and S^* is a convergent scheme.*

Proof Recall that Theorem 6 applies to the i-mean analogue of a linear rule, because it can be seen as a special case of the log-exp construction, and use that that $\|S^*\| < N$ implies that S is a convergent scheme. □

Corollary 12 *If S has positive mask and T is the i-mean analogue of S in a CH manifold, then limit curves $T^\infty p$ enjoy C^2 smoothness for all input data, provided $\|S^*\| < N$, and both S^*, S^{**} are convergent schemes.*

7.2 Different ways of subdivision in matrix groups.

Before we return to Pos_n, we dicuss a minor point which arises from the fact that the construction of symmetric spaces G/H, if H is taken as the trivial group $H = \{e\}$, leads back to G itself. It follows that now two different kinds of log/exp subdivision rules can be defined in the group G: one construction with applies to groups, and another one which applies to symmetric spaces.

The symmetric space construction with $G = G/\{e\}$ leads to (using the terminology of Section 2) $s = \mathrm{id}$, $\mathfrak{s} = \mathfrak{g}$, and $\mathfrak{k} = 0$. This yields the functions $p \oplus v = p\exp(p^{-1}v)$ and $q \ominus p = p\log(p^{-1}q)$.

On the other hand, log-exponential subdivision for Lie groups has already been studied (see [4]), using the operators $p \mathbin{\widehat{\oplus}} v = p\exp v$ and $q \mathbin{\widehat{\ominus}} p = \log(p^{-1}q)$.

However, it is obvious that $p_i \oplus \sum_j \alpha_j(p_j \ominus p_i) = p_i \mathbin{\widehat{\oplus}} \sum_j \alpha_j(p_j \mathbin{\widehat{\ominus}} p_i)$, so the log-exponential analogues of subdivision rules constructed with either method are the same. We may therefore speak of *the* log-exponential analogue of a linear rule S in a group G.

7.3 Subdivision in Pos_n as subset of a matrix group.

As Pos_n is a subset of the matrix group GL_n it is interesting to compare subdivision rules defined in GL_n with subdivision rules of Pos_n. By coincidence, the operator $\oplus$ in Pos_n reads $p \oplus w = p\exp(p^{-1}w)$ (by Example 5), which is the same as the $\oplus$ operator in the symmetric space $\mathrm{GL}_n/\{e\}$. This fact is stated as follows:

Proposition 13 *Assume that T and T' are the log-exp analogues of a linear subdivision rule S in the space Pos_n and in the group GL_n, respectively. Then $T = T'|_{\mathrm{Pos}_n}$. This is also true for geodesic analogues.*

This result is important because it allows us to transfer known properties of Lie group subdivision schemes to Pos_n. We give only one example in the form of a corollary of a theorem of [18]:

Corollary 14 *If the linear subdivision rule S of Equation (7) is interpolatory with dilation factor 2 and has Hölder smoothness γ, then its log/exp analogue in Pos_n defined by $Tp_{2i} = p_i$ and $Tp_{2i+1} = p_i \oplus \sum_{j\in\mathbb{Z}} a_{1+2(i-j)}(p_j \ominus p_i)$ has the same property.*

7.4 Subdivision with linear subspaces.

Here we briefly describe how the Grassmann manifold $G_{d,n}$ of d-dimensional linear subspaces of $\mathbb{R}^n$ is made a symmetric space and how to find geodesics. The special orthogonal group SO_n is acting on $X = G_{d,n}$, and we choose as base point the subspace L spanned by the first d canonical basis vectors. We use block matrix notation and write $n \times n$ skew-symmetric matrices $v \in \mathfrak{so}_n$ in the form $\left(\begin{smallmatrix} U & P \\ -P^T & W \end{smallmatrix}\right)$, where $U \in \mathfrak{so}_d$ and $W \in \mathfrak{so}_{n-d}$. The (infinitesimal) rotations which leave the base point invariant are exactly those which transform both L and $L^\perp$ within themselves, so $\mathfrak{k}$ is characterized by $P = 0$, and the subgroup K is the direct product of rotations within L and rotations within $L^\perp$. We can therefore write $G_{d,n} = \mathrm{SO}_n/\mathrm{SO}_d \times \mathrm{SO}_{n-d}$.

The reflection s defined by $\left(\begin{smallmatrix} U & P \\ -P^T & W \end{smallmatrix}\right) \stackrel{s}{\longmapsto} \left(\begin{smallmatrix} U & -P \\ P^T & W \end{smallmatrix}\right)$, makes $G_{d,n}$ a symmetric space (infinitesimal version) with $+1$ eigenspace $\mathfrak{k}$ and -1 eigenspace $\mathfrak{s} = \{\left(\begin{smallmatrix} 0 & P \\ -P^T & 0 \end{smallmatrix}\right)\}$. We observe that with the reflection ρ_L in the subspace L we have

$s(x) = \rho_L x \rho_L$. It follows that the involutive automorphism $\sigma(x) = \rho_L \circ x \circ \rho_L$ of the group SO_n, whose fixed point set is K, has differential s. Consequently $G_{d,n}$ is a symmetric space also in the narrower sense.

For a description of geodesics in $G_{d,n}$, it is sufficient to consider geodesics emanating from L. Consider the following special infinitesimal rotation

$$v = \left(\begin{array}{cc|cc} 0_m & & & \\ & 0_{d-m} & \mathrm{diag}(\alpha_1,\dots,\alpha_{d-m}) & \\ \hline & -\mathrm{diag}(\alpha_1,\dots,\alpha_{d-m}) & 0_{d-m} & \\ & & & 0_{n-2d+m} \end{array}\right) \in \mathfrak{s}. \text{ Then}$$

$$\exp(tv) = \left(\begin{array}{cc|cc} E_m & & & \\ & \mathrm{diag}(\cos(t\alpha_j)) & \mathrm{diag}(\sin(t\alpha_j)) & \\ \hline & -\mathrm{diag}(\sin(t\alpha_j)) & \mathrm{diag}(\cos(t\alpha_j)) & \\ & & & E_{n-2d+m} \end{array}\right) \in \mathrm{SO}_n,$$

and $\exp(tv)(L)$ is a geodesic. The geometric meaning of this is that L undergoes $d-m$ independent rotations in mutually orthogonal planes until it reaches some space $L' = \exp(t_0 v)(L)$. These planes are spanned by ON bases $\{e_{m+j}, e_{d+j}\}$ for $j = 1,\dots,d-m$. Vectors $e_1,\dots,e_m$ span the intersection $L \cap L'$, and the remaining $n-2d+m$ basis vectors span the orthogonal complement $(L+L')^\perp$ of the spaces of interest. If, for a given subspace L', we can find a change of coordinates such that $e_1,\dots,e_d$ still span L and L' is spanned by the first d columns of the matrix v above, then $t \mapsto \exp(tv)(L)$ is a geodesic connecting L with L'. Representing tangent vectors by elements of $\mathfrak{s}$ we then have $L \oplus v = L'$ and $L' \ominus L = v$.

It is well known how to find these planes and basis vectors: With the orthogonal projections p, p' onto L, L', consider the mapping $p \circ p'|L$. Obviously $L \cap L'$ is an eigenspace for the eigenvalue 1, and we assume it to be spanned by $e_1,\dots,e_m$. Each of the remaining eigenvalues $\cos^2\alpha_j$ ($j = 1,\dots,d-m$) corresponds to an eigenvector e_{m+j} of $p \circ p'$ in L.

— In case of $\cos\alpha_j \neq 0$, Gram-Schmidt orthonormalization applied to $e_{m+j}, p'(e_{m+j})$ yields the basis $\{e_{m+j}, e_{d+j}\}$ of the corresponding plane of rotation. The angle α_j is uniquely determined by the requirement $0 < \alpha_j < \frac{\pi}{2}$, and it is not difficult to see that even with multiple eigenvalues, v does not depend on the choice of eigenvector.

— Zero eigenvalues $\cos\alpha_j$ occur if part of L is orthogonal to L' (and vice versa). In this case we choose an appropriate number of basis vectors e_{m+j} in the null space $L \cap (L')^\perp$, and corresponding basis vectors e_{d+j} in $L' \cap L^\perp$. It is no longer possible to choose v in a both invariant and unique way.

7.5 Subdivision with affine subspaces.

The set $G^{\mathrm{aff}}_{d,n}$ of d-dimensional affine subspaces of $\mathbb{R}^n$ is acted upon by the Euclidean motion group SE_n. If $d \neq 0$ it does not carry an invariant Riemannian metric, and except for the *line space* $G^{\mathrm{aff}}_{1,3}$ not even an indefinite one. The latter's geodesics have been used for instance for the design of freeform ruled surfaces [12].

We demonstrate why $X = G^{\mathrm{aff}}_{d,n}$ is a symmetric space: The group $G = \mathrm{SE}_n$ is a matrix group, consisting of elements $g = \left(\begin{smallmatrix} 1 & 0 \\ a & A \end{smallmatrix}\right)$, where $a \in \mathbb{R}^n$, and $A \in \mathrm{SO}_n$. It acts on a point of $\mathbb{R}^n$ via $g \circ x = Ax + a$, and $g \circ (h \circ x) = (g \cdot h) \circ x$. The Lie algebra of SE_n is called $\mathfrak{se}_n$: It consists of $\left(\begin{smallmatrix} 0 & 0 \\ v & V \end{smallmatrix}\right)$ where $v \in \mathbb{R}^n$, $V \in \mathfrak{so}_n$ ($\mathfrak{so}_n$ being the $n \times n$ skew-symmetric matrices).

We choose as a base point in X the d-dim. subspace L which contains 0 and is spanned by the first d canonical basis vectors. Its stabilizer K is generated by translations along K and rotations which leave L (and $L^\perp$) invariant. The Lie algebra $\mathfrak{k}$ consequently is spanned by infinitesimal translations along L, and infinitesimal rotations within L and $L^\perp$, which means that $v \in L$ and V in block matrix notation reads $V = \left(\begin{smallmatrix} U & P \\ -P^T & W \end{smallmatrix}\right)$, with $P = 0$, $U \in \mathfrak{so}_d$, $W \in \mathfrak{so}_{n-d}$. Define the reflection s and the subspace $\mathfrak{s}$ by

$$\begin{pmatrix} 0 & 0 & 0 \\ u & U & P \\ w & -P^T & W \end{pmatrix} \stackrel{s}{\longmapsto} \begin{pmatrix} 0 & 0 & 0 \\ u & U & -P \\ -w & P^T & W \end{pmatrix} \implies \mathfrak{s} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & P \\ w & -P^T & 0 \end{pmatrix} \right\}.$$

Like in the previous section, we consider the reflection ρ_L in the base subspace. It is easy to see that s is the differential of the involution $\sigma(x) = \rho_L \circ x \circ \rho_L$ which is defined in the group SE_n. Therefore, $G^{\mathrm{aff}}_{d,n} = \mathrm{SE}_n/K$ is indeed a symmetric space, both in the infinitesimal and the narrower sense.

For the special case $G^{\mathrm{aff}}_{1,3}$ we describe how to determine the geodesic which connects lines L_1, L_2, i.e., how to find v such that $L_1 \oplus v = L_2$. The geodesic we look for is the curve $c(t) = L_1 \oplus (tv)$. By invariance with respect to Euclidean motions we way use a Cartesian coordinate system such that L_1 is the x axis, the z axis is the common orthogonal transversal of L_1, L_2, and L_2 contains the point $(0, 0, a)$ and is parallel to $(\cos\phi, \sin\phi, 0)$. Now L_1 is the base point, so we only must find $\left(\begin{smallmatrix} 0 & 0 \\ v & V \end{smallmatrix}\right) \in \mathfrak{s}$ such that $\exp(\left(\begin{smallmatrix} 0 & 0 \\ v & V \end{smallmatrix}\right)) \circ L_1 = L_2$. The following choice apparently works:

$$\exp\left(t \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\phi & 0 \\ 0 & \phi & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t\phi) & -\sin(t\phi) & 0 \\ 0 & \sin(t\phi) & \cos(t\phi) & 0 \\ a & 0 & 0 & 1 \end{pmatrix}.$$

This yields the result that in line space, the geodesic which connects two given lines corresponds to a helical motion whose axis is the common transversal of the given lines. For general values of d, n the construction is similar and leads to a combination of translations and rotations in mutually orthogonal planes like in Section 7.4. For $d = 0$, the geodesics are the straight lines of $\mathbb{R}^n$, and for $d = n - 1$ (hyperplanes) the geodesic connecting L_1, L_2 corresponds to a rotation about the axis $L_1 \cap L_2$.

8 Concluding remarks

In recent years, quite a number of results concerning the convergence and smoothness of subdivision rules which apply to dense enough data have been obtained by the method of proximity (where [15] is a prototype). In contrast to this, convergence for *all* input data has not been explored to a great extent so far. In fact a more detailed analysis in the spirit of Theorem 5 as well as the obvious extensions to the multivariate case are the topic of a forthcoming paper.

Acknowledgments.

E. Nava Yazdani's work on this paper was done at TU Graz, where he was supported by grant No. 18575 of the Austrian Science Fund (FWF). A preliminary version of this paper which did not contain Section 4 appeared as geometry preprint 2007/06, August 2007, at TU Graz. Andreas Weinmann gratefully acknowledges support by FWF grant No. P19780.

References

1. N. Dyn, P. Grohs, and J. Wallner. Approximation order of interpolatory nonlinear subdivision schemes. *J. Comput. Appl. Math.*, 2009. to appear.
2. P. Grohs. Higher order smoothness of interpolatory multivariate subdivision in Lie groups. *IMA J. Numer. Anal.*, 27:760–772, 2009.
3. P. Grohs. Smoothness equivalence properties of univariate subdivision schemes and their projection analogues. *Num. Math.*, 113:163–180, 2009.
4. P. Grohs and J. Wallner. Log-exponential analogues of univariate subdivision schemes in Lie groups and their smoothness properties. In M. Neamtu and L. L. Schumaker, editors, *Approximation Theory XII: San Antonio 2007*, pages 181–190. Nashboro Press, 2008.
5. S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Academic Press, 1978.
6. H. Karcher. Riemannian center of mass and mollifier smoothing. *Comm. Pure Appl. Math*, 30:509–541, 1977.
7. W. S. Kendall. Probability, convexity, and harmonic maps with small image. I. Uniqueness and fine existence. *Proc. London Math. Soc. (3)*, 61:371–406, 1990.
8. S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. II*. Wiley, 1969.
9. S. Lang. *Fundamentals of Differential Geometry*. Springer, 1999.
10. M. Moakher. A differential geometric approach to the geometric mean of symmetric positive-definite matrices. *SIAM J. Matrix Anal. Appl.*, 26(3):735–747 (electronic), 2005.
11. L. Noakes. Non-linear corner cutting. *Adv. Comp. Math.*, 8:165–177, 1998.
12. K. Sprott and B. Ravani. Kinematic generation of ruled surfaces. *Adv. Comp. Math.*, 17:115–133, 2002.
13. I. Ur Rahman, I. Drori, V. C. Stodden, D. L. Donoho, and P. Schröder. Multiscale representations for manifold-valued data. *Multiscale Mod. Sim.*, 4:1201–1232, 2005.
14. J. Wallner. Smoothness analysis of subdivision schemes by proximity. *Constr. Approx.*, 24:289–318, 2006.
15. J. Wallner and N. Dyn. Convergence and C^1 analysis of subdivision schemes on manifolds by proximity. *Comput. Aided Geom. Des.*, 22:593–622, 2005.
16. G. Xie and T. P.-Y. Yu. Smoothness equivalence properties of manifold-valued data subdivision schemes based on the projection approach. *SIAM J. Numer. Anal.*, 45:1200–1225, 2007.
17. G. Xie and T. P.-Y. Yu. Smoothness equivalence properties of general manifold-valued data subdivision schemes. *Multiscale Mod. Sim*, 7:1073–1100, 2008.
18. G. Xie and T. P.-Y. Yu. Smoothness equivalence properties of interpolatory Lie group subdivision schemes. *IMA J. Numer. Anal.*, 2009. to appear.